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Resummation of $(\beta_0 \alpha_s)^n$ Corrections in QCD: Techniques and Applications to the τ Hadronic Width and the Heavy Quark Pole Mass

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Abstract:

We propose to resum exactly any number of one-loop vacuum polarization insertions into the scale of the coupling of lowest order radiative corrections. This makes maximal use of the information contained in one-loop perturbative corrections combined with the one-loop running of the effective coupling and provides a natural extension of the familiar BLM scale-fixing prescription to all orders in the perturbation theory. It is suggested that the remaining radiative corrections should be reduced after resummation. In this paper we implement this resummation by a dispersion technique and indicate a possible generalization to incorporate two-loop evolution. We investigate in some detail higher order perturbative corrections to the τ decay width and the pole mass of a heavy quark. We find that these corrections tend to reduce $\alpha_s(m_\tau)$ determined from τ decays by approximately 10% and increase the difference between the bottom pole and $\overline{\text{MS}}$ -renormalized mass by 30%.

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1 Introduction

During the past decade QCD has turned from qualitative descriptions to quantitative predictions for the strong interactions. The influx of more accurate experimental data represents a constant stimulus to improve theoretical predictions. While an understanding of confinement remains as evasive as ever, these predictions rely on the validity of nonperturbative factorization in processes governed by a large momentum scale. This allows to isolate the presently uncalculable infrared dynamics in a set of universal parameters or functions. One is then led to the conclusion that relations of physical quantities can be calculated in perturbation theory. For certain observables, such as those derived from totally inclusive processes with no hadrons in the initial state, one obtains parameter-free predictions (except for the strong coupling) to logarithmic accuracy in the hard scale.

Much effort has been invested in the computation of higher order QCD radiative corrections. For a few observables for which quark masses are unimportant, second order corrections in the strong coupling α_s are known. Third order expressions exist for hadroproduction in e^+e^- -annihilation and τ -lepton decays and for the Gross-Llewellyn-Smith and Bjorken sum rule in deep inelastic scattering. On the other hand, observables involving quark masses, for example decay widths of hadrons containing a heavy quark, are known only to first order. In most cases, the extension of present results is not simply a question of algebraic complexity and algorithms for handling it. The existing calculations have exploited presently known techniques to the frontier beyond which new methods need to be designed, a task which might not be completed soon. In this situation it is interesting to explore possible sources of systematically large corrections in higher orders, which, once identified, might then be taken into account exactly to all orders. In doing so, one may hope to reduce the uncertainty due to ignorance of exact higher order coefficients.

Resummations of this type are familiar and necessary for many problems involving disparate mass scales. Large perturbative corrections in higher orders are associated with a logarithm of the ratio of these scales and can often be summed with renormalization group techniques, which allow to obtain the dominant power of that logarithm to all orders by one-loop calculations. Although in practice it is not always clear whether the logarithms dominate the constant pieces – in particular, since the coefficients of logarithms grow geometrically with order of α_s , whereas the constants grow factorially –, such a resummation is controlled by an “external” parameter (the logarithm of two scales) that can be varied, at least in fictitious limits. For the problem at hand, we assume that such renormalization group improvement has already been done or consider observables that depend only on a single scale. We are thus interested in systematically large contributions to constant terms with no external scale at our disposal.

The estimation of constant terms of yet uncalculated coefficients, which is closely related to the problem of scale- and scheme-setting for truncated perturbative expansions, is often thought to attempt the impossible. While indeed for any particular observable it is impossible to assess with rigour the quality of a certain scale-setting prescription, some prescriptions may be supported by general physical arguments and turn out to be closer to exact results *a posteriori*. The most distinguished prescription

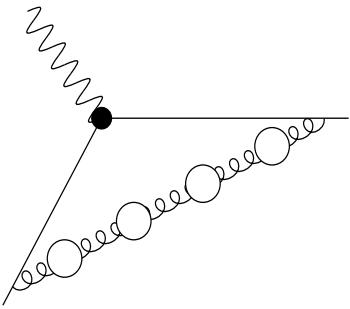


Figure 1: A typical diagram with multiple fermion loop insertions into the lowest order correction to a generic physical quantity.

of this kind has been formulated by Brodsky, Lepage and Mackenzie [1]. Observing that in QED all scale-dependence of the coupling results from photon vacuum polarization, they suggested that the effect of fermion loop insertions into a photon line in higher orders be absorbed into the scale of the coupling of a given order and proposed to apply this criterion to QCD as well. In practical applications, this suggestion has mainly been realized in second order in α_s . To this accuracy, only a single fermion loop insertion needs to be calculated and the relevant contribution can be traced by its dependence on the number of light fermions N_f . In general, at order α_s^{n+1} , the effect of one-loop evolution of the coupling can still be obtained by the highest power of N_f in the flavour-dependence of coefficients. The highest power of N_f originates solely from diagrams with n fermion loop insertions, such as in Fig. 1, which are much easier to calculate than flavour-independent terms. In QCD, the identification of contributions related to one-loop renormalization of the coupling implies that the coefficient of the highest power in N_f should in fact not multiply N_f to some power, but the combination $N_f - 33/2$, as it appears in the Gell-Mann-Low function to leading order. Computing diagrams as in Fig. 1 and then replacing N_f by $N_f - 33/2$, one takes into account partial contributions from other diagrams, which are much harder to evaluate exactly (the precise identification of these contributions with particular diagrams is not straightforward and gauge-dependent). It turns out that in all cases where comparison with exact second order results is possible, this replacement approximates the exact coefficient amazingly well in the $\overline{\text{MS}}$ -scheme [2]. In what follows we shall refer as “Naive Nonabelianization” (NNA) [3] to the hypothesis that a substantial part of higher order radiative corrections can be accounted for by running of the coupling, in the sense of substitution of $N_f \rightarrow N_f - 33/2$ in the term with the highest power of N_f . We should mention that BLM scale-setting and NNA are technically identical and the distinction is rather a matter of interpretation: The BLM scale-setting by its physical motivation does not necessarily claim that genuine higher order corrections missed by the above substitution should be small, whereas NNA assumes this stronger assertion.

This paper is concerned with an exposition of details of the recent proposal [4] of

two of us to extend the BLM procedure to higher orders and, in particular, to sum all contributions associated with one-loop running of the coupling. We emphasize that this procedure has repeatedly been suggested in the past [1, 5], but has apparently not been pursued to the point of practical implementation. A summation similar to, but technically different from [4] was also presented in [6]. We believe that this resummation is useful, because the effect of evolution of the coupling is a systematic source of potentially large perturbative coefficients. In such a situation it is advantageous to incorporate this effect in theoretical predictions even if it transcends a fixed-order perturbative approximation.

To become definite, we use the following convention for the QCD β -function¹:

$$\begin{aligned} \mu^2 \frac{d\alpha_s}{d\mu^2} &= \beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \dots ; \\ \beta_0 &= -\frac{1}{4\pi} \left(11 - \frac{2}{3} N_f \right), \quad \beta_1 = -\frac{1}{(4\pi)^2} \left(102 - \frac{38}{3} N_f \right). \end{aligned} \quad (1.1)$$

For a generic physical observable R (which for the sake of illustration we assume to depend only on a single, large scale Q), we may eliminate the N_f -dependence in radiative corrections in favour of the β -function coefficients β_0, β_1 etc. For now (and most parts of this paper) we restrict ourselves to one-loop running, induced by β_0 . The (truncated) perturbative expansion of R is written as

$$\begin{aligned} R - R_{tree} &= \sum_{n=0}^N [r_{n0} + r_{n1} N_f + \dots + r_{nn} N_f^n] \alpha_s(Q)^{n+1} \\ &= r_0 \alpha_s(Q) \sum_{n=0}^N [\delta_n + d_n (-\beta_0)^n] \alpha_s(Q)^n, \end{aligned} \quad (1.2)$$

where

$$d_n = (-6\pi)^n \frac{r_{nn}}{r_0}. \quad (1.3)$$

Let us stress again that d_n is unambiguously determined by diagrams with n insertions of fermion bubbles as in Fig. 1 and only δ_n requires a true $n+1$ -loop calculation. The δ_n ($n \geq 2$) could be further separated into contributions from two-loop evolution and the effect of one-loop evolution on the genuine two-loop corrections (see [7] for δ_2). We introduce

$$M_N(-\beta_0 \alpha(Q)) = 1 + \sum_{n=1}^N d_n (-\beta_0 \alpha(Q))^n \quad (1.4)$$

¹Since we discuss corrections proportional to $(-\beta_0 \alpha_s)^n$, the reader accustomed to the normalization $11 - (2/3)N_f$ may think of an expansion in terms of $\alpha_s/(4\pi)$.

as a measure of how much the lowest order correction is modified by summing N one-loop vacuum polarization insertions. To make an explicit connection to BLM scale-setting, we define

$$\alpha_s(Q_N^*) = \alpha_s(Q) M_N(-\beta_0 \alpha_s(Q)) \quad (1.5)$$

absorbing the effect of vacuum polarization in the scale Q_N^* of the coupling. For $N = \infty$, $\alpha_s(Q^*)$ ($Q^* \equiv Q_\infty^*$) is transparently interpreted as the running coupling averaged over the lowest order corrections. Isolating the integration over gluon virtuality in the lowest-order diagram, we may write

$$r_0 \alpha_s(Q) = \alpha_s(Q) \int d^4k F(k, Q) \frac{1}{k^2}. \quad (1.6)$$

Then

$$r_0 \alpha_s(Q^*) = \frac{r_0 \alpha_s(Q)}{1 - \beta_0 \alpha_s(Q) \ln(Q^{*2}/Q^2)} = \int d^4k F(k, Q) \frac{\alpha_s(k \exp[C/2])}{k^2}, \quad (1.7)$$

where C is a the scheme-dependent subtraction constant for the fermion loop. The difference between the leading-order BLM scale Q_1^* and Q^* is precisely the difference between averaging the running coupling itself, or the logarithm of its argument, over the lowest-order diagram. The values of Q^* and Q_N^* for $N > 1$ depend on the value of $\alpha_s(Q)$. Such a dependence is intrinsic to any generalization of the BLM prescription beyond leading-order and has previously been noted in [8, 7]. As a result we may express any physical quantity as

$$R - R_{tree} = r_0 \left(\alpha(Q) M_\infty(-\beta_0 \alpha_s(Q)) + \sum_{n=1}^{\infty} \delta_n \alpha_s(Q)^{n+1} \right). \quad (1.8)$$

Separating radiative corrections in this way and summing a partial set of them into $M_\infty(-\beta_0 \alpha_s)$ can be motivated in different ways. First of all, as already mentioned, this procedure is supported empirically at the two-loop level. Provided the three-loop coefficient is sizeable, one might again expect that a significant portion of it can already be accounted for by $M_2(-\beta_0 \alpha_s)$ and, in higher orders, by $M_N(-\beta_0 \alpha_s)$.

It must be noted that such a statement depends on the choice of scheme and, since the δ_n depend on N_f for $n > 1$, also on N_f for a given scheme. Although Q^* is defined such that $\alpha_s(Q^*)$ is scheme-independent if one consistently uses the one-loop β -function, an arbitrary N_f -independent finite renormalization can always ruin the supposed dominance of $\beta_0^n \alpha_s^{n+1}$ -terms by implying a definition of the coupling that leads to anomalously large values of the remaining terms δ_n . This remark applies of course to any partial resummation, whether it is based on a systematic (external) parameter or not as in the case at hand². In this respect a resummation utilizing one-

²We do not consider β_0 on the same footing as, say, $\ln Q^2/m^2$, since it is an intrinsic parameter of QCD. However, we shall somewhat loosely refer to the approximation of ignoring the δ_n as the “large- β_0 limit”.

loop running is similar to resummation of π^2 's that arise from analytic continuation from the space-like to the time-like region of, for instance, the Sudakov form factor [9] or the coupling as in e^+e^- -annihilation or τ decays [10], and, in fact, comprises the resummation for τ decays as a special case. Since empirically the β_0 -term dominates second order radiative corrections for many observables in the $\overline{\text{MS}}$ scheme, one would not expect the resummation of $\beta_0\alpha_s^{n+1}$ -terms to be numerically useful in schemes that differ from $\overline{\text{MS}}$ by a redefinition of the coupling that changes significantly the relative importance of flavour-dependent and flavour-independent terms.

The dominance of $\beta_0^n\alpha_s^{n+1}$ -terms is lent further support from the behaviour of perturbative coefficients in large orders (renormalons). It is precisely the effect of vacuum polarization, which leads to the expectation that in sufficiently large orders, the coefficients should indeed have the form

$$r_n \sim K (a\beta_0)^n n! n^b. \quad (1.9)$$

The series is thus divergent and, provided it is asymptotic at all, cannot approximate the true result to any desired accuracy, even if the exact coefficients could be computed to arbitrary order. The evaluation of diagrams as in Fig. 1 provides some insight into this ultimate limitation of perturbative QCD and into the approach to the asymptotic limit. In this paper, however, we are not so much interested in large orders and effects suppressed by powers of the hard scale Q . Summation of effects associated with one-loop running of the coupling should prove useful in intermediate orders (say, $n = 2 - 6$), when perturbation theory is still reliable. It is amusing to speculate whether the empirical dominance of β_0 at the two-loop level indicates that one is already close to the asymptotic behaviour. The latter is derived from a saddle point expansion and the large contribution arises from internal momenta either much smaller or much larger than the external scale Q . Proximity to the asymptotic behaviour – if true – would indicate that the distribution of internal momenta is already well-approximated by a Gaussian even though the main contribution to the Feynman integral is not from small or large momenta.

The consideration of large orders provides also useful guidance to the limitations that arise from the restriction to one-loop evolution effects. As a matter of fact, diagrams associated with one-loop evolution do not provide the correct constants b and K in the large- n behaviour. At large n , the effect of two-loop evolution on a single gluon line and one-loop evolution on two gluon lines becomes equally important and should eventually be taken into account³. In practical applications, we will often find that the series has to be truncated due to its divergence at rather low n . Thus, many diagrams which are required to establish the formal large- n behaviour never become relevant. Moreover, provided δ_1 is already not large, one should also expect that vacuum polarization insertions into two gluon lines will remain comparatively small.

Let us repeat that the resummation of $\beta_0^n\alpha_s^{n+1}$ -terms can only be judged *a posteriori* and can fail to provide a good estimate of higher order corrections for any particular

³ For example, an infinite number of two-loop insertions is necessary to obtain the correct value of b [11].

quantity. Even in this case, we believe that this resummation is physically motivated and that it is appropriate to absorb this particular class of higher order corrections into the scale of the coupling in lowest order. Application of BLM scale-setting to leading order (absorbing the contribution from one fermion loop) yields $Q_1^* = Q e^{-d_1/2}$. Upon re-expansion of $\alpha_s(Q_1^*)$ to one-loop accuracy for the β -function, this pretends a geometric growth $d_n^{\text{BLM}} = d_1^n$ to be compared with the factorial growth of the exact coefficients. Resummation to all orders corrects for this discrepancy by adjusting Q_∞^* such that the expansion of $\alpha_s(Q_\infty^*)$ gives the correct values of d_n . Thus to the accuracy of one-loop evolution the result of resummation is equivalently expressed as $\alpha_s(Q) M_\infty(-\beta_0 \alpha_s)$ and we often prefer this form of presentation. It is no longer equivalent to $\alpha(Q_\infty^*)$, if one adopts two-loop accuracy for α_s . Since for large n the true d_n will always outgrow d_1^n , one might conclude that the usual BLM scale-setting underestimates higher order radiative corrections. In practice, the effect is often just the opposite: The most important contributions come from intermediate n and it turns out that in many cases d_1 is comparatively large (Q_1^* is small), so that $d_1^n > d_n$ in this region. The usual BLM scale-setting therefore typically overestimates the size of radiative corrections associated with one-loop running.

The remainder of the paper is organized as follows: In Sect. 2 we develop in detail the technique to implement the resummation. We use a dispersion technique to reduce the problem to a one-dimensional integral over lowest order radiative corrections computed with finite gluon mass which is suitable to numerical evaluation. Thus compared to the complications of a complete higher order calculation, this resummation can be performed with little computational expense. Compared to a similar implementation of the standard BLM scale-setting [12], the computation of higher order coefficients d_n comes with no additional expense at all. It is convenient to introduce the Borel transform as a generating function of higher order radiative corrections. The principal value definition of the Borel integral serves as a starting point for the summation of the series within a certain accuracy, indicated by the presence of infrared renormalons. We shall also see that this definition requires all kinematic invariants to be large compared to Λ_{QCD} , reflecting the inapplicability of perturbation theory as a starting point for summation, if this requirement is not met. In this Section we also generalize the results of [4] to quantities with anomalous dimension and include quarks with finite masses in the loops.

In Sect. 3 we apply the resummation to the hadronic width of the τ lepton and find a 10% decrease of $\alpha_s(m_\tau)$ due to four- and higher loop corrections. We discuss the possibility of $1/m_\tau^2$ -corrections in the light of our resummation. In general we prefer to be agnostic about power corrections and stick to perturbation theory. An exception to this rule is that we do not want to introduce power corrections in conflict with the operator product expansion in euclidian space without good reasons (which we do not have). Under this assumption we show that principal value resummation does not introduce $1/m_\tau^2$ -terms to the decay width, provided the coupling is chosen appropriately. Sect. 4 contains a detailed discussion of the resummation of $(-\beta_0)^n \alpha_s^{n+1}$ -terms for the pole mass of a heavy quark. We combine the resummation with the exact two-loop result to give an estimate of the difference between the pole mass and the $\overline{\text{MS}}$ -renormalized mass. We keep finite quark masses inside loops which allows to trace the origin of large

coefficients to the relevant regions of internal momentum.

In Sect. 5 we formulate one possible extension of our resummation to incorporate partially the effect of two-loop running resulting in $\beta_1\beta_0^{n-2}$ -corrections. This extension is again guided by the flavour-dependence of coefficients and can be considered as an extension of recent work by Brodsky and Lu [7]. The size of corrections is illustrated for the vector correlation function relevant to τ decays and for the pole mass of a heavy quark. Sect. 6 contains conclusions. Three Appendices deal with technical issues: In Appendix A we derive a simple expression for subtractions needed for logarithmically ultraviolet divergent quantities and collect the details of results presented in Sect. 2. Appendix B contains analytic formulae for the lowest order radiative corrections to $R_{e^+e^-}$ and hadronic τ decays with finite gluon mass. In Appendix C we list exact results for some abelian five-loop diagrams to the vector correlation function.

In a companion paper [13] we shall discuss the implications of resummation for heavy quark decays and the determination of $|V_{bc}|$.

2 Techniques

The aim of this Section is to develop a systematic approach to the calculation of diagrams with an arbitrary number of fermion loop insertions, such as in Fig. 1. We assume a generic physical (short-distance) quantity R and a renormalization scheme that does not introduce an artificial N_f dependence, such as $\overline{\text{MS}}$. It is also assumed that lowest order radiative corrections to R do not involve the gluon self-coupling. Subtracting the tree-level contribution, we are left with the (truncated) perturbative expansion

$$R - R_{tree} = \sum_{n=0}^N r_n \alpha_s^{n+1}. \quad (2.1)$$

The coefficients r_n are polynomials in N_f ,

$$r_n = r_{n0} + r_{n1}N_f + \dots + r_{nn}N_f^n, \quad (2.2)$$

and we will calculate the coefficients r_{nn} , which originate from n fermion loop insertions into the lowest-order diagram. We then write

$$r_n = r_0 [\delta_n + (-\beta_0)^n d_n], \quad (2.3)$$

where $d_n = (-6\pi)^n r_{nn}/r_0$ absorbs the term with the largest power of N_f . The effect of one-loop evolution of the coupling on lowest order radiative corrections is then entirely contained in the d_n and in the remainder of the paper we will consider the δ_n as corrections to the approximation of “Naive Nonabelianization”. As a measure of how much the lowest order radiative correction is modified by including N vacuum polarization insertions, we define

$$\begin{aligned} M_N(-\beta_0 \alpha_s) &\equiv 1 + \sum_{n=1}^N d_n (-\beta_0 \alpha_s)^n, \\ M_\infty(-\beta_0 \alpha_s) &\equiv M_{N \rightarrow \infty}(-\beta_0 \alpha_s). \end{aligned} \quad (2.4)$$

Taken literally, the limit $N \rightarrow \infty$ does not exist, because the series diverges. It will be interpreted in the sense of Eq. (2.8) below. For further use, we introduce the shorthand notation

$$a_s(\mu) = -\beta_0 \alpha_s(\mu), \quad (2.5)$$

where μ is the normalization scale, which often will be suppressed for brevity. In this Section we are concerned only with technical aspects of the calculation of $M_N(a_s)$ and $M_\infty(a_s)$.

2.1 Borel transform vs. finite gluon mass

A convenient way to deal with diagrams with multiple loop insertions is to calculate the Borel transform

$$B[R](u) \equiv \sum_n \frac{r_n}{n!} (-\beta_0)^{-n} u^n, \quad (2.6)$$

which can be used as a generating function for the fixed-order coefficients [14]:

$$r_n = (-\beta_0)^n \frac{d^n}{du^n} B[R](u)_{|u=0}. \quad (2.7)$$

Another advantage of presenting the results in form of the Borel transform is that the result for the sum of all diagrams can easily be recovered by the integral representation

$$r_0 M_\infty(a_s) = \frac{1}{a_s} \int_0^\infty du e^{-u/a_s} B[R](u), \quad (2.8)$$

where the integration goes over positive values of the Borel parameter u . Note that we define the Borel parameter u with an additional factor $(-\beta_0)$ compared to the conventional definition. In fact, Eq. (2.8) requires some explanation: As it stands the integral is not defined, because the Borel transform generally has singularities on the integration path, known as infrared renormalons. We shall adopt a definition of the integral based on deforming the contour above or below the real axis or on a principal value prescription. These prescriptions are not unique and their difference, which is exponentially small in the coupling, must be considered as an uncertainty which can not be removed within perturbation theory. This will be discussed in more detail. A second question concerns the existence of the principal value integral and the behaviour of the Borel transform at $u = \infty$. If we consider a physical quantity that depends only on a single

scale Q , then, to one-loop running accuracy, renormalization group invariance entails that the Borel transform can be written as $(\mu^2/Q^2)^u$ times a Q - and μ -independent function $F(u)$. Combining the factor $(\mu^2/Q^2)^u$ with $e^{-u/a_s(\mu)}$ in Eq. (2.8), we deduce that the principal value integral exists, provided that Q^2 is sufficiently large compared to Λ_{QCD}^2 and $F(u)$ does not increase faster than any exponential as u approaches positive infinity. Since the second property is satisfied in all examples which we shall meet (and we may conjecture that this is generally true for the Borel transform computed from higher order corrections due to vacuum polarization), we shall assume in general that all kinematic invariants on which R depends explicitly are sufficiently large compared to Λ_{QCD}^2 (to be precise, the difference of two invariants is also an invariant). This is not an additional assumption needed to take the Borel integral. Without it perturbative methods are not applicable and there is no perturbation theory to start with. In particular, we do not know whether the Borel transform, which is useful in connection with short-distance expansions, can serve as a *bona fide* starting point to summation in the strong coupling regime⁴.

In simple cases the Borel transform can be calculated directly. This is due to the fact that the evaluation of diagrams with multiple fermion bubble insertions in Landau gauge corresponds to the evaluation of the lowest-order diagram with the effective propagator

$$D^{AB}(k) = i\delta^{AB} \frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{k^4} \frac{1}{1 + \Pi(k^2)}, \quad (2.9)$$

where

$$\Pi(k^2) = a_s \ln \left(-\frac{k^2}{\mu^2} e^C \right) \quad (2.10)$$

and C is a scheme-dependent finite renormalization constant. In the $\overline{\text{MS}}$ -scheme $C = -5/3$, in the V-scheme [1] $C = 0$.

For a chain of fermion loops contributing to a physical amplitude with euclidian external momenta, one can separate the integration over the gluon momenta to write it as

$$r_0 \alpha_s(\mu) M_\infty(a_s) = \int d^4 k F(k, Q) \frac{1}{k^2} \frac{\alpha_s(\mu)}{1 + \Pi(k^2)}, \quad (2.11)$$

where the transverse projector that appears in the gluon propagator in Landau gauge is assumed to be included in the function $F(k, Q)$ and Q stands for a collection of kinematic invariants.

A crucial simplification arises since for diagrams with only one fermion bubble chain the Borel transformation applies to the expansion in α_s of the propagator in Eq. (2.9) itself, rather than to the set of diagrams as a whole. The effective (Borel-transformed) gluon propagator is [14]:

⁴If some kinematic invariants are small, one might still be able to define the Borel integral as an analytic continuation. However, in this regime all power corrections in a short-distance expansion are of the same importance and the analytic continuation is useless unless the summation of the short-distance expansion is understood.

$$B[\alpha_s D_{\mu\nu}^{AB}(k)](u) = i\delta^{AB} \left(\frac{e^C}{\mu^2}\right)^{-u} \frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{(-k^2)^{2+u}}. \quad (2.12)$$

Thus, the task of calculating the Borel transform of Feynman diagrams with bubble insertions reduces to the calculation of the leading-order diagram with the usual gluon propagator raised to an arbitrary power, which is familiar from analytic regularization [15]. This trick suffices to derive an expression for the Borel transform of the polarization operator with light quarks [14, 16, 17], of the heavy quark self-energy [17] and several more complicated cases in connection with heavy quark expansions, which can be found in [17, 18, 19, 3].

However, in most phenomenologically interesting cases an analytic expression for the Borel transform is difficult to obtain, especially for observables involving more than one scale. Even if the exact Borel transform is obtainable, taking n derivatives to evaluate fixed-order coefficients (see Eq. (2.7)) may turn out to be a complicated task. In this paper we shall work out a different technique, which extracts the desired information on higher orders from the lowest-order diagrams, calculated with a finite gluon mass [4, 12]. Calculations of one-loop diagrams with finite gauge boson mass have become routine for electroweak radiative corrections, which allows to hope that this technique is generally applicable to a wide range of observables in QCD. In this way one obtains a concise integral representation for the Borel transform.

For a while let us restrict our discussion to euclidian quantities, which are not sensitive to the gluon self-coupling to leading order.⁵ Call $r_0(\lambda^2)$ the leading-order radiative correction calculated with finite gluon mass λ and $r_0 \equiv r_0(0)$. To be precise, we define $r_0(\lambda^2)$ as the sum of all Feynman diagrams to leading order, which in general may be ultraviolet divergent and need to be renormalized. In this Section we restrict our discussion to cases where no explicit renormalization is needed, which is the case, e.g. for the derivative of the polarization operator in Eq. (3.1), or transition amplitudes related to heavy quark decays with on-shell mass renormalization. It is easy to show that this assumption is equivalent to the requirement that $r_0(\lambda^2)$ vanishes as $\lambda^2 \rightarrow \infty$. Equally, in the case of Borel representation for the diagrams with fermion bubbles, we assume that fermion loops are renormalized, and no additional explicit renormalization is necessary. This assumption can easily be relaxed. A detailed discussion of renormalization is given in Appendix A, where we work out the missing overall subtractions for the general case.

We keep the standard gauge-fixing and work with the propagator

$$-i\delta^{AB} \frac{1}{k^2 - \lambda^2 + i\epsilon} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 - \xi\lambda^2 + i\epsilon} \right]. \quad (2.13)$$

The relation to the Borel transform of the massless propagator in Eq. (2.12) (in Landau gauge, $\xi = 0$, or Feynman gauge, $\xi = 1$) is established by an (inverse) Mellin representation

⁵With the latter restriction, we do not have problems with gauge invariance, which otherwise prohibits introduction of a finite gluon mass, unless QCD is embedded, for example, in an $SU(3)$ Higgs model.

$$\frac{1}{k^2 - \lambda^2} = \frac{1}{2\pi i} \frac{1}{k^2} \int_{-1/2-i\infty}^{-1/2+i\infty} du \Gamma(-u) \Gamma(1+u) \left(-\frac{\lambda^2}{k^2}\right)^u. \quad (2.14)$$

Writing down the leading-order radiative correction with non-vanishing gluon mass λ (in Landau gauge) as

$$r_0(\lambda^2) = \int d^4 k F(k, Q) \frac{1}{k^2 - \lambda^2}. \quad (2.15)$$

and comparing to Eq. (2.11), one finds the identity [20]

$$r_0(\lambda) = \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} du \Gamma(-u) \Gamma(1+u) \left(\frac{\lambda^2}{\mu^2} e^C\right)^u B[R](u) \quad (2.16)$$

Taking the inverse, we get [4]

$$\begin{aligned} B[R](u) &= -\frac{\sin(\pi u)}{\pi} \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left(\frac{\lambda^2}{\mu^2} e^C\right)^{-u} [r_0(\lambda^2) - r_0(0)] \\ &= -\frac{\sin(\pi u)}{\pi u} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} e^C\right)^{-u} r'_0(\lambda^2) \end{aligned} \quad (2.17)$$

where $r'_0(\lambda^2) = (d/d\lambda^2)r_0(\lambda^2)$. If $r_0(\lambda^2) \sim \lambda^{2a}$ for small λ^2 , the first line exists, when $0 < u < a$. The second line provides the analytic continuation to an interval about $u = 0$, so that derivatives at zero can be taken. Thus fixed-order coefficients r_n can be expressed in terms of the integrals

$$J_k \equiv \int_0^\infty d\lambda^2 \ln^k(\lambda^2/\mu^2) r'_0(\lambda^2) \quad k \leq n. \quad (2.18)$$

For instance (cf. Eqs. (2.3) and (2.7)),

$$\begin{aligned} d_0 &= \frac{1}{r_0} [-J_0] = 1 \\ d_1 &= \frac{1}{r_0} [J_1 + C J_0] \\ d_2 &= \frac{1}{r_0} \left[-J_2 - 2C J_1 - \left(C^2 - \frac{\pi^2}{3}\right) J_0 \right] \\ d_3 &= \frac{1}{r_0} \left[J_3 + 3C J_2 + (3C^2 - \pi^2) J_1 + (C^3 - C\pi^2) J_0 \right] \end{aligned} \quad (2.19)$$

etc. and $r_n = r_0 (-\beta_0)^n d_n$. r_1 coincides with the result of Smith and Voloshin [12] (who use a scheme with $C = 0$), integrating of their expression by parts.

Next, we evaluate the Borel integral in Eq. (2.8) to obtain a compact answer for the sum of diagrams with fermion bubble insertions to arbitrary order, up to the ambiguities caused by renormalons as mentioned below Eq. (2.8). For the following derivation we assume that the Borel integral is defined with a contour slightly above the positive real axis. We also recall that this integral exists, if all kinematic invariants are large compared to Λ_{QCD}^2 , which we assume. To find a representation for the so-defined sum in terms of an integral over λ^2 , one needs to insert Eq. (2.17) in Eq. (2.8) and take the u -integral explicitly. The u -integration is elementary, but the interchange of orders of integration in u and λ^2 can not be done immediately, because the λ^2 -integral in Eq. (2.17) is not defined for all u on the integration path. It is convenient to use the first representation in Eq. (2.17) and write it as⁶

$$\begin{aligned} r_0 M_\infty(a_s) &= -\frac{1}{2\pi i} \int_0^\infty du e^{-u/a_s} \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left(\frac{\lambda^2 e^{-i\pi}}{\mu^2} e^C \right)^{-u} [r_0(\lambda^2) - r_0(0)] \\ &\quad + \frac{1}{2\pi i} \int_0^\infty du e^{-u/a_s} \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left(\frac{\lambda^2 e^{i\pi}}{\mu^2} e^C \right)^{-u} [r_0(\lambda^2) - r_0(0)] \end{aligned} \quad (2.20)$$

The integration over u in the first of the two terms above can easily be taken, rotating the contour to the positive imaginary axis. The quarter-circle at infinity does not contribute, because of the behaviour of the Borel-transform at infinity as discussed above. The λ^2 -integral exists for positive imaginary u and the order of integrations can be interchanged. The u -integral in the second term would equally easily be taken by rotating the contour to the negative imaginary axis, but the infrared renormalon singularities on the real axis obstruct this deformation. It can only be done at the price of picking up the residues from the infinite number of poles. A more elegant solution is to rotate first the integration contour over λ^2 to the second sheet: $\lambda^2 \rightarrow \lambda^2 e^{-i(\pi+\epsilon)}$. Here we note that for euclidian quantities, there are no singularities of the λ^2 -integrand in the lower complex plane. Next, the u -integration can again be performed by rotation to the positive imaginary axis, as above. This integration gives

$$\int_0^{i\infty} du \exp\{-u[1/a_s + \ln(|\lambda^2|/\mu^2 e^C)] - i\epsilon\} = \frac{a_s}{1 + a_s \ln(|\lambda^2|/\mu^2 e^C) - i\epsilon} \quad (2.21)$$

and introduces a pole singularity, located at

$$\lambda_L^2 = -\mu^2 \exp[-1/a_s - C], \quad (2.22)$$

which is simply the position of the Landau pole in the running coupling (in the V-scheme). Now one can rotate the λ^2 -integral back from the second sheet to the real

⁶The separation of the two terms in the following equation reintroduces the pole at $u = 0$ in each of the two λ^2 -integrals. The pole cancels in the sum of both terms, and both terms can be manipulated separately regardless of this pole. This can be justified as follows: One splits the λ^2 -integral before separation into the two terms at some sufficiently large λ_T^2 . The contribution from λ_T^2 to infinity can be handled without difficulty. For the integral form 0 to λ_T^2 one can proceed as following Eq. (2.20) and both terms have no singularity at $u = 0$.

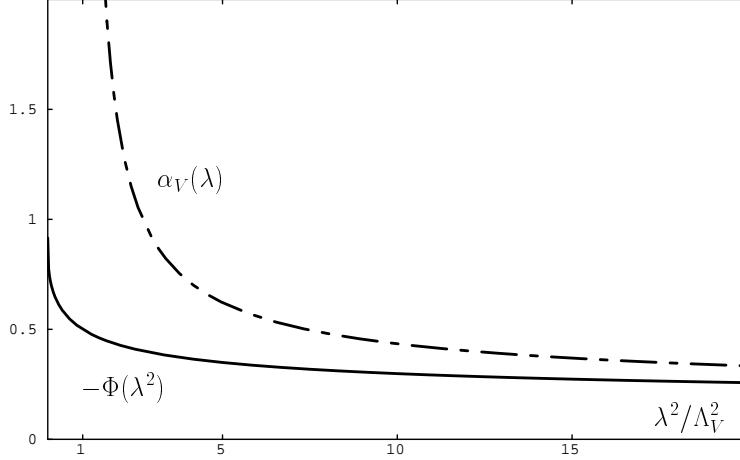


Figure 2: One-loop running coupling in the V-scheme, $C = 0$, (broken line) and effective coupling $-\Phi(\lambda^2)$ (solid line) as functions of λ^2/Λ_V^2 .

positive axis. In this way one encounters the pole in Eq. (2.21), whose residue has to be added. Collecting all the terms, we get after some algebra:

$$r_0 M_\infty(a_s) = - \int_0^\infty \frac{d\lambda^2}{\lambda^2} \frac{a_s}{|1 + a_s \ln(-\lambda^2/\mu^2 e^C)|^2} [r_0(\lambda^2) - r_0(0)] + \frac{1}{a_s} [r_0(\lambda_L^2 - i\epsilon) - r_0(0)] \quad (2.23)$$

where the $i\epsilon$ prescription corresponds to defining the Borel integral above the real axis: The contours are rotated in the opposite directions if the Borel integral is defined below the real axis, with the only modification that the sign of the $i\epsilon$ -prescription is reversed in the result. Finally, integrating by parts in the first term, we obtain

$$r_0 a_s(\mu) M_\infty(a_s(\mu)) = \int_0^\infty d\lambda^2 \Phi(\lambda^2) r'_0(\lambda^2) + [r_0(\lambda_L^2 - i\epsilon) - r_0(0)] \quad (2.24)$$

where

$$\Phi(\lambda^2) = -\frac{1}{\pi} \arctan \left[\frac{a_s(\mu)\pi}{1 + a_s(\mu) \ln(\lambda^2/\mu^2 e^C)} \right] - \Theta(-\lambda_L^2 - \lambda^2), \quad (2.25)$$

which coincides with the result of [4], obtained by a different method. Note that the term with the Θ -function exactly cancels the jump of the arctan at $\lambda^2 = -\lambda_L^2$.

Eq. (2.24) presents the desired answer for the sum of diagrams with any number of fermion bubbles in terms of an integral over the gluon mass. This relation is one of the main technical tools which we suggest in this paper, and we discuss its structure in detail.

First, notice that Eq. (2.24) has a very intuitive and simple interpretation: The quantity $r'_0(\lambda^2)$ (with certain reservations) can be considered as the contribution to the integral from gluons of virtuality of order λ^2 , and the function $-\Phi(\lambda^2)$ can be understood as an effective charge. At large scales, $-\Phi(\lambda^2)$ essentially coincides with $\alpha_V(\lambda)$, the QCD coupling in the V-scheme [1], but in distinction to it remains finite at small λ^2 , see Fig. 2. The absence of a Landau pole in this effective coupling implies that the integral is a well-defined number and the fact that we have started with the expression in Eq. (2.8) that is ill-defined due to infrared renormalon singularities (equivalently, attempted to sum a non-Borel summable series) is isolated in the Landau pole contribution $r_0(\lambda_L^2)$. Whenever infrared renormalons are present, $r_0(\lambda^2)$ develops a cut at negative λ^2 . The real part of the above expression for the bubble sum coincides with the principal value of the Borel integral and, in particular, coincides with the Borel sum, when it exists. The imaginary part provided by $r_0(\lambda_L^2)$ coincides with the imaginary part of the Borel integral, when the contour is deformed above (or below) the positive real axis. We therefore adopt the real part of Eq. (2.24) as a definition of $M_\infty(a_s)$, and consider the imaginary part (divided by π) as an estimate of the intrinsic uncertainty from summing a (non-Borel summable) divergent series without any additional nonperturbative input. Note that this imaginary part is proportional to a power of $\lambda_L^2/Q^2 \sim \exp(-1/a_s(Q))$ and therefore is suppressed by powers of Λ_{QCD}/Q .

Second, notice that the result in Eq. (2.24) is manifestly scale- and scheme-invariant, provided the running of the coupling is consistently implemented with the one-loop β -function: $a_s(\mu_1) = a_s(\mu_2)/(1 + a_s(\mu_2) \ln(\mu_1^2/\mu_2^2))$. In particular, $\alpha_s(Q^*)$ is formally independent of the finite renormalization C for the fermion loop. To be precise, in schemes that do not introduce artificial flavour dependence, the coefficients of the expansion that relates the couplings in two different schemes have itself an expansion of the type of Eq. (2.2). Within the restriction to fermion loops with subsequent restoration of β_0 one must again keep only the highest power of N_f in these coefficients. If couplings are related in this way, the numerical value for $\alpha_s(Q^*) = \alpha_s(Q)M_\infty(a_s(Q))$ is scheme-independent.

In practice, working with the full QCD β -function one introduces a scheme-dependence in higher orders in $1/N_f$ similarly as usually with higher orders in $\alpha_s(Q)$. The quality of the NNA as compared to exact coefficients depends on the numerical magnitude of terms that are formally suppressed by powers of N_f (or β_0) and therefore depends on the scheme. Since empirically the success of NNA in low orders is observed in the $\overline{\text{MS}}$ scheme, we cannot expect it to hold in schemes that differ from $\overline{\text{MS}}$ by terms that are formally of higher order but numerically large. Thus, although the resummation accounts exactly for the contribution of fermion loops in any scheme, we believe that its phenomenological relevance is tied to the $\overline{\text{MS}}$ scheme, or others that are “reasonable” in the above sense.

Third, we remark that Eq. (2.24) applies without modification to quantities like inclusive decay rates, which can be obtained starting from a suitable amplitude in euclidian space and taking the total imaginary part upon analytic continuation to Minkowski space. The structure of the λ^2 -integral remains unaffected, and it is only the quantity $r'_0(\lambda^2)$ that should be substituted by the corresponding decay rate calculated with finite gluon mass (For heavy quark decays, no explicit renormalization is needed, when the

decay rates are expressed in terms of pole masses.). Similarly, Eq. (2.24) is applicable to various semi-inclusive quantities, with the only formal restriction that a weight-function chosen to specify the final state does not resolve quark-antiquark pairs in fermion bubbles, that is, their phase space must be completely integrated. It is not applicable to quantities like hadronic event shapes.

The derivation of Eq. (2.24) is only slightly modified, when $r_0(\lambda^2)$ represents a physical cross section. In this case, $r_0(\lambda^2)$ is written as a sum of virtual and real gluon emission, where real gluon emission occurs only when λ^2 is below a certain threshold $\lambda_T(Q)^2$,

$$r_0(\lambda^2) = r_{virt}(\lambda^2) + r_{real}(\lambda^2) \Theta(\lambda_T(Q)^2 - \lambda^2). \quad (2.26)$$

When all kinematic invariants, represented by Q , are large compared to Λ_{QCD}^2 , the same is true for $\lambda_T(Q)^2$. It is simple to show that Eq. (2.17) holds unmodified for the Borel transform of a cross section with Eq. (2.26). When deriving Eq. (2.24) one splits the λ^2 -integral in Eq. (2.17) at $\lambda_T(Q)^2$. The contribution from 0 to $\lambda_T(Q)^2$ can be treated in the same way as before (adding and subtracting a contribution from a semicircle with radius $\lambda_T(Q)^2$). For the contribution from $\lambda_T(Q)^2$ to infinity, one may interchange the order of λ^2 - and u -integrals directly, since the first exists for all u . Then, the u -integral can be taken, since $\lambda_T(Q)^2$ is large compared to Λ_{QCD}^2 (and therefore larger than $-\lambda_L^2$). After the u -integral is taken, we combine both pieces. The final result is identical to Eq. (2.24).

We should also mention that the Borel transform of physical cross sections generically contains a factor $\sin \pi u$, which seems to obstruct the rotation of the u -integral to the imaginary axis. Closer inspection of Eq. (2.20) shows that the two separate λ^2 -integrals there correspond in fact to the Borel transform of

$$e^{\pm i\pi u} \frac{B[R](u)}{\sin \pi u}, \quad (2.27)$$

which is sufficient to guarantee convergence at the imaginary u relevant to the two terms in Eq. (2.20).

2.2 Quantities involving renormalization

So far the derivation was restricted to quantities, where the lowest order radiative correction r_0 is ultraviolet (and infrared) finite. Then, in diagrams involving insertion of fermion loops into the gluon line, counterterms are needed only for the fermion loop subdiagrams, but no overall subtraction for the diagram itself (after summation of all diagrams that contribute to r_n) is required. We consider now the more general case, that the quantity $R(Q)$ has an anomalous dimension. One can still use Eq. (2.9), but the resulting Borel transform is singular at $u = 0$. This singularity is compensated by adding a function $S_R(u)$, which accounts for the missing counterterms. The function $S_R(u)$ in the $\overline{\text{MS}}$ scheme is given in Eq. (A.7).

When expressed in terms of the lowest order radiative correction with non-zero gluon mass, the necessity of additional subtractions is reflected in a divergence of the λ^2 -integral at large λ^2 for $u = 0$ in the second line of Eq. (2.17), since $r_0(\lambda^2)$ grows logarithmically at large λ^2 (We assume a logarithmic ultraviolet divergence.). The additional counterterms not associated with fermion loop insertions amount to the subtraction of the leading term in the large- λ^2 expansion of $r_0(\lambda^2)$ and to the addition of some scheme specific contributions. In the $\overline{\text{MS}}$ scheme, Eqs. (2.17) and (2.24) are replaced by ($C = -5/3$)

$$\begin{aligned} B[R](u) &= -\frac{\sin \pi u}{\pi u} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} \left[r'_0(\lambda^2) - \frac{r_\infty}{\lambda^2} \Theta(\lambda^2 - \mu^2 e^{-C}) \right] \\ &\quad + \frac{1}{u} \left(\tilde{G}_0(u) - r_\infty \frac{\sin \pi u}{\pi u} \right), \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} r_0 a_s M_\infty(a_s) &= \int_0^\infty d\lambda^2 \Phi(\lambda^2) \left(r'_0(\lambda^2) - \frac{r_\infty}{\lambda^2} \Theta(\lambda^2 - \mu^2 e^{-C}) \right) + [r_0(\lambda_L^2) - r_0(0)] \\ &\quad + \int_0^{a_s} \frac{du}{u} (G_0(u) - r_\infty) + r_\infty \left[\frac{\arctan(\pi a_s)}{\pi a_s} + \frac{1}{2} \ln \left(1 + \pi^2 a_s^2 \right) - 1 \right]. \end{aligned} \quad (2.29)$$

Here, as always, $a_s = a_s(\mu)$. The derivation of this result together with the definition of all new quantities that appear in the above equation is given in Appendix A. In particular, $G_0(u)$ is essentially the anomalous dimension of R (to leading order in $1/N_f$).

2.3 Quarks with finite masses

In practical applications it can be important to trace the number of active fermion flavours, which may effectively decrease in high orders because important integration regions shift towards decreasing momenta, if the series is dominated by infrared renormalons in large or intermediate orders. Thus, it is worthwhile to generalize the above technique to include massive quarks in fermion loops. For definiteness, we shall consider here the case of one massive, and $N_f - 1$ exactly massless flavours. The generalization to several massive flavours is then obvious. β_0 is to be taken with N_f flavours, including the massive one, since we have in mind a situation, where the quark mass is non-zero, but smaller than the external momenta Q . We study the decoupling of quarks with finite masses in higher orders on a particular example in Sect. 4.

The vacuum polarization $\Pi(k^2)$ in Eq. (2.9) includes summation over flavours. With one massive quark of mass m , it is modified to

$$\begin{aligned} \Pi(k^2) &= a_s \left[\ln(-k^2/\mu^2) + C - \Delta(k^2, m^2) \right], \\ \Delta(k^2, m^2) &= \frac{1}{6\pi\beta_0} \int_0^\infty \frac{ds}{s - k^2} [\rho(s, m^2) - 1], \end{aligned} \quad (2.30)$$

where

$$\rho(s, m^2) = \left(1 + \frac{2m^2}{s}\right) \sqrt{1 - \frac{4m^2}{s}} \Theta(s - 4m^2). \quad (2.31)$$

Performing the integral gives

$$\begin{aligned} \Delta(k^2, m^2) &= \frac{1}{6\pi\beta_0} \left\{ \ln\left(\frac{-k^2}{m^2}\right) + \frac{4m^2}{k^2} - \sqrt{1 - \frac{4m^2}{k^2}} \left(1 + \frac{2m^2}{k^2}\right) \right. \\ &\quad \left. \times \left[\ln\left(\frac{-k^2}{m^2}\right) + \ln\left[\frac{1}{2} \left(1 - \frac{2m^2}{k^2} + \sqrt{1 - \frac{4m^2}{k^2}}\right)\right] \right] \right\}. \end{aligned} \quad (2.32)$$

To derive the expression for the sum of fermion loop insertions, we follow the method of [4, 12] and substitute the effective propagator $(1 + \Pi(k^2))^{-1}$ in Eq. (2.11) by the dispersion relation

$$\frac{1}{1 + \Pi(k^2)} = \frac{1}{\pi} \int_0^\infty d\lambda^2 \frac{1}{k^2 - \lambda^2} \frac{\text{Im } \Pi(\lambda^2)}{|1 + \Pi(\lambda^2)|^2} + \frac{1}{k^2 - \lambda_L^2} \frac{1}{\Pi'(\lambda_L^2)} \quad (2.33)$$

where $\Pi'(\lambda^2) \equiv d/d\lambda^2 \Pi(\lambda^2)$ and $\lambda_L^2 < 0$ is the solution of

$$1 + \Pi(\lambda_L^2) = 0 \quad (2.34)$$

provided it exists. If no solution exists, the second term in Eq. (2.33) is absent. We now use

$$\frac{1}{\pi} \text{Im } \Pi(\lambda^2) = -a_s \left[1 + \frac{1}{6\pi\beta_0} (\rho(\lambda^2, m^2) - 1) \right] \quad (2.35)$$

and write

$$\frac{1}{k^2 - \lambda^2} = \frac{1}{\lambda^2} \left(\frac{k^2}{k^2 - \lambda^2} - 1 \right). \quad (2.36)$$

Interchanging the order of integrations in k and λ^2 , we arrive at

$$r_0 M_\infty(a_s) = -a_s \int_{-\infty}^\infty \frac{d\lambda^2}{\lambda^2} \frac{r_0(\lambda^2) - r_0(0)}{|1 + \Pi(\lambda^2)|^2} \left[1 + \frac{1}{6\pi\beta_0} (\rho(\lambda^2, m^2) - 1) \right] + \frac{r_0(\lambda_L^2) - r_0(0)}{\lambda_L^2 \Pi'(\lambda_L^2)} \quad (2.37)$$

which coincides with Eq. (2.23) in the limit $m \rightarrow 0$.

A representation of the Borel transform which allows the calculation of fixed-order perturbative coefficients is obtained in a similar way. Starting from Eq. (2.11), one only needs

$$\text{Im } B \left[\frac{\alpha_s}{1 + \Pi(\lambda^2)} \right] (u) = \text{Im } e^{-u\Pi(\lambda^2)/a_s} = e^{-u\text{Re}\Pi(\lambda^2)/a_s} \sin \left[-u\text{Im}\Pi(\lambda^2)/a_s \right]. \quad (2.38)$$

Then we proceed as in the massless case and get

$$\begin{aligned} B[R](u) &= -\frac{1}{\pi} \int_0^\infty \frac{d\lambda^2}{\lambda^2} [r_0(\lambda^2) - r_0(0)] \sin \left\{ u\pi \left[1 + \frac{1}{6\pi\beta_0} (\rho(\lambda^2, m^2) - 1) \right] \right\} \\ &\quad \times \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} \exp \left\{ u\text{Re}\Delta(\lambda^2, m^2) \right\}. \end{aligned} \quad (2.39)$$

For $m^2 = 0$ we recover the old result in Eq. (2.17). To avoid possible bad behaviour at $\lambda^2 \rightarrow \infty$ and to separate the mass dependence, we add and subtract the expression for $m^2 = 0$. Defining

$$T(u, \lambda^2, m^2) = \sin \left\{ u\pi \left[1 + \frac{1}{6\pi\beta_0} (\rho(\lambda^2, m^2) - 1) \right] \right\} \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} \exp \left\{ u\text{Re}\Delta(\lambda^2, m^2) \right\}, \quad (2.40)$$

we obtain, finally

$$\begin{aligned} B[R](u) &= -\frac{\sin(\pi u)}{\pi u} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} r'_0(\lambda^2) \\ &\quad - \frac{1}{\pi} \int_0^\infty \frac{d\lambda^2}{\lambda^2} [r_0(\lambda^2) - r_0(0)] [T(u, \lambda^2, m^2) - T(u, \lambda^2, 0)]. \end{aligned} \quad (2.41)$$

The second line gives the correction due to finite quark masses inside loops. We derived this result for quantities that do not need renormalization. However, in the $\overline{\text{MS}}$ scheme subtractions are mass-independent. Therefore, only the first term in Eq. (2.41) is affected by subtractions, and in precisely the same way as for $m^2 = 0$ (cf. Sect. 2.2).

We should point out that Eq. (2.37) for the sum of fermion loops has been derived through a dispersion relation. For massless quarks, comparison of the derivation of Eq. (2.24) with the one in [4] shows that the result obtained by a dispersion relation coincides with the principal value prescription of the Borel integral. We do not offer such an equivalence for massive quarks and there is reason to doubt that it is true in minimal subtraction schemes for the fermion loops. To illustrate this, suppose there were only a single massive particle inside loops, which produces a negative contribution to the β -function. For any finite value of the mass of this particle, the vacuum polarization behaves as k^2/m^2 at very small virtuality k . Therefore the factorial growth of coefficients from the infrared region of integration is eliminated at very large order. There are no infrared renormalon poles and therefore no ambiguity in the Borel transform (though it may be sharply peaked at those values, where singularities appear as m approaches zero). On the other hand $1/(1 + \Pi(k^2))$ does have a pole in the euclidian,

when m is smaller than a critical value and $\Pi(k^2)$ is defined by minimal subtraction. This leads to an ambiguity in the second term in Eq. (2.37). The critical value is of order Λ_{QCD} and the discrepancy will be noticeable only for such small quark masses. However, for quark masses of order Λ_{QCD} , the quark mass must be considered as an infrared regulator and one can no longer straightforwardly identify an infrared parameter, such as the gluon condensate, with non-analytic terms in a finite gluon mass. As is well-known, the gluon condensate must be redefined to absorb the non-analyticities in light quark masses as well. This is indicated by the highly singular behaviour of the Borel transform as the quark mass goes to zero. In our practical application the mentioned discrepancy is numerically irrelevant, and we do not pursue this point further. Notice that such a difficulty does not appear in finite order perturbative coefficients, derived from Eq. (2.41), and affects only the Landau pole term, in which we are mainly interested as a measure of intrinsic uncertainty.

2.4 Renormalon ambiguities and extended Bloch-Nordsieck cancellations

We want to emphasize the close relation of renormalon singularities to non-analytic terms in the expansion of lowest-order radiative corrections in powers of the gluon mass [20]. A formal relation between singularities in the Borel plane and non-analytic terms in λ^2 is established by Eq. (2.16). Each non-analytic term proportional to $(\sqrt{\lambda^2})^{2n+1}$ in the expansion of $r_0(\lambda^2)$ at small λ^2 is in one-to-one correspondence to a single-pole singularity of $B[R](u)$ at positive half-integers $u = n + 1/2$. Each non-analytic term proportional to $\lambda^{2n} \ln \lambda^2$ corresponds to a single pole at positive integer $u = n$. Likewise, non-analytic terms in the expansion at large λ^2 of type $\sqrt{\lambda^2}^{(-2n-1)}$ or $\lambda^{-2n} \ln \lambda^2$ correspond to a single-pole singularities of the Borel transform at negative $u = -n - 1/2$ and $u = -n$, respectively. For double or higher poles, higher powers of $\ln \lambda^2$ appear.

We recall that through the presence of singularities for real positive values of the Borel parameter the perturbation series signals its deficiency: Explicit non-perturbative corrections must be added to make the full answer unambiguous. In fact, the infrared renormalon problem is just one manifestation of a generally accepted wisdom: Perturbative calculations are only reliable if the essential integration regions include momenta much larger than Λ_{QCD} . The above relation between infrared renormalons and non-analytic terms in the expansion in powers of the infrared regulator like the gluon mass is thus natural and expected. A comment is necessary, however, to explain why only *non-analytic* terms in the expansion at small λ^2 are related to the infrared behaviour, and simple power-like terms, λ^{2n} , are not.

A small gluon mass $\lambda^2 \sim \Lambda_{\text{QCD}}^2$ not only eliminates contributions of small momenta $k^2 \sim \Lambda_{\text{QCD}}^2$, but also modifies the gluon propagator at virtualities of order $k^2 \sim Q^2$. In this region of momenta $1/(k^2 - \lambda^2)$ can be expanded and produces (infrared insensitive) corrections of the form $\lambda^2/k^2 \sim \lambda^2/Q^2$. In perturbative calculations these terms are unimportant, since the corrections are suppressed by powers of Q^2 . Hence the common practice to use the finite gluon mass as an IR regulator in calculations of various QCD observables. The famous Bloch-Nordsieck cancellations guarantee that terms propor-

tional to $\ln \lambda^2$, which appear at intermediate steps of the calculation, cancel in final answers for inclusive quantities. Calculations aiming at *power-like* accuracy must trace accurately the fate of power-like terms in the gluon mass. Since analytic terms λ^{2n} come entirely from the expansion of the gluon propagator at large virtualities of order Q^2 , they disappear when the regulator is removed. Only non-analytic terms are related to the infrared behaviour, and indicate the failure of perturbation theory to account for this region properly. Thus, tracing non-analytic terms in the expansion of leading-order radiative correction at small gluon masses, one can trace the sensitivity of particular quantities to the IR region, and, in particular, judge upon existence of non-perturbative corrections suppressed by particular powers of Q^2 .

Stated otherwise, the absence of particular power-suppressed corrections in physical quantities can be understood as an extension of the Bloch-Nordsieck cancellations. For example, the fact that in Wilson's operator product expansion for the correlation function $\Pi(Q^2)$ the $1/Q^2$ -corrections are absent, corresponds in this language to the cancellation of corrections of order $\lambda^2 \ln \lambda^2$ in the correlation function. Note once more that analytic terms proportional to λ^2 do not necessarily cancel, and in fact they are present in a quantity, closely related to $\Pi(Q^2)$, the τ -lepton hadronic width, see below. In turn, the existence of a gluon condensate contribution, $\langle G^2 \rangle / Q^4$, implies that contributions of order $\lambda^4 \ln \lambda^2$ do not cancel.

3 Hadronic τ decays

In this Section we apply the summation of one-loop running coupling effects developed above to observables related by analyticity to the correlation function of vector (axial-vector) currents

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(Q^2) = i \int d^4x e^{iqx} \langle 0 | T\{j_\mu^\dagger(x) j_\nu(0)\} | 0 \rangle, \quad Q^2 = -q^2. \quad (3.1)$$

Quantities of prime physical interest related to $\Pi(Q^2)$ are the cross section of e^+e^- annihilation

$$R_{e^+e^-}(s) \equiv \frac{\Gamma(e^+e^- \rightarrow \text{hadrons})}{\Gamma(e^+e^- \rightarrow \mu^+\mu^-)} = 12\pi \text{Im } \Pi_V(s) \quad (3.2)$$

(neglecting Z -boson exchange) and the τ -lepton total hadronic width

$$\begin{aligned} R_\tau &\equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)} \\ &= 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im } \Pi^{(1)}(s) + \text{Im } \Pi^{(0)}(s) \right], \end{aligned} \quad (3.3)$$

where $s = q^2$ and we introduced the decomposition

$$\Pi_{V/A}^{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi_{V/A}^{(1)}(q^2) + q_\mu q_\nu \Pi_{V/A}^{(0)}(q^2) \quad (3.4)$$

for the vector and axial-vector correlation function and defined $\Pi^{(i)}(s) = \Pi_V^{(i)}(s) + \Pi_A^{(i)}(s)$. For the purpose of our discussion we neglect the strange quark mass and omit the overall CKM factor $|V_{ud}|^2 + |V_{us}|^2 \approx 1$.

The *exact* (nonperturbative) correlation functions should be analytic in the complex s -plane cut along the positive axis. Exploiting this property, we may transform Eq. (3.3) into [21]

$$R_\tau = 6\pi i \int_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(1)}(s) + \Pi^{(0)}(s) \right]. \quad (3.5)$$

The same analytic property holds to any *finite* order in perturbation theory in $\alpha_s(\mu)$ (although the discontinuity is arbitrarily wrong at small s). Eqs. (3.3) and (3.5) are equivalent if the correlation functions are substituted either by the exact values or finite order perturbative expansions. In addition, in perturbation theory, the vector and axial-vector contributions coincide. The equivalence of Eqs. (3.3) and (3.5) does not hold in renormalization group improved perturbation theory or after fermion loop summation. This will be discussed extensively below.

In what follows we concentrate on τ decays, which despite the low energy scale involved, are considered to provide one of the most reliable determinations of the QCD coupling [22]. The state-of-the-art perturbative calculations yield [23] in the $\overline{\text{MS}}$ scheme:

$$\begin{aligned} R_\tau = & 3 \left[1 + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right) + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right)^2 (6.3399 - 0.3792N_f) \right. \\ & \left. + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right)^3 (48.5832 - 7.8795N_f + 0.1579N_f^2) \right] + O(\alpha_s^4). \end{aligned} \quad (3.6)$$

The leading non-perturbative corrections due to contributions of local operators of dimension 6 have been estimated [21, 22] and turn out to be small, below 1% compared to the tree-level unity in brackets above. Thus, the principal uncertainty of the determination of α_s from the τ hadronic width comes from unknown higher-order terms in the perturbative expansion.

The purpose of this Section is twofold: We analyze summation of $(-\beta_0\alpha_s)^n$ perturbative corrections and conclude – with certain *caveats* – that the cumulative effect of higher order corrections beyond order α_s^3 is somewhat larger than the exact order α_s^3 -correction. Resummation of vacuum polarization reduces the value of $\alpha_s(m_\tau)$ extracted from τ decays by about 10%. Second, we endeavour to clarify and speculate on a conceptual point: Several authors [24, 25, 22] considered (with different conclusions) the possibility that power corrections proportional to $1/m_\tau^2$ may creep into R_τ from various sources (summation of large orders in perturbation theory – infrared and ultraviolet renormalons –, freezing of a physical coupling in the infrared, violations of duality), whose absence in the approach reviewed above is crucial to ascertain its power

for the determination of α_s . We elaborate on two aspects, which are sometimes omitted from the discussion: The necessity to define a coupling parameter to power-like (in $1/m_\tau$) accuracy and the distinction between hypothetical $1/m_\tau^2$ -corrections to the OPE of correlation functions in the euclidian and to the τ decay width itself.

3.1 Higher orders in $(-\beta_0) \alpha_s$

To start with, let us demonstrate that Naive Nonabelianization provides indeed an excellent approximation in low orders. The NNA approximation is obtained from Eq. (3.6) keeping the term with the leading power of N_f only, and restoring the full β_0 by the substitution $N_f \rightarrow N_f - 33/2$. The result is

$$\begin{aligned} R_\tau^{NNA} = & 3 \left[1 + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right) + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right)^2 (6.2568 - 0.3792N_f) \right. \\ & \left. + \left(\frac{\alpha_s(m_\tau^2)}{\pi} \right)^3 (42.9883 - 5.2107N_f + 0.1579N_f^2) \right] + O(\alpha_s^4). \end{aligned} \quad (3.7)$$

The accuracy is impressive: For the practical case $N_f = 3$ NNA predicts a coefficient 28.7773 for the $(\alpha_s/\pi)^3$ correction, to be compared to the exact coefficient 26.3658. We should mention, however, that this coincidence is also slightly misleading, since a substantial part of the exact coefficient is from a combination of β -function coefficients and lower order coefficients of the correlation function $\Pi(s)$ and results from contour integration in Eq. (3.5).

To estimate higher orders, we recall that for the polarization operator, Eq. (3.1), an analytic expression is available for the Borel transform of the sum of diagrams with fermion loop insertions [14, 16, 26, 17]. To get rid of an (irrelevant) overall subtraction, it is convenient to take one derivative with respect to Q^2 and to consider

$$D(Q^2) \equiv Q^2 \frac{d\Pi(Q^2)}{dQ^2}. \quad (3.8)$$

Then (the simple representation quoted here is adapted from [16], see [26, 17])

$$B[D(Q^2)](u) = \frac{8}{3\pi^3} \left(\frac{Q^2}{\mu^2} e^C \right)^{-u} \frac{u}{1 - (1-u)^2} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - (1-u)^2)^2} \quad (3.9)$$

and [26]

$$B[R_\tau](u) = 12\pi^2 B[D(Q^2)](u) \sin(\pi u) \left[\frac{1}{\pi u} + \frac{2}{\pi(1-u)} - \frac{2}{\pi(3-u)} + \frac{1}{\pi(4-u)} \right] \quad (3.10)$$

Taking derivatives of $B[R_\tau](u)$ (see Eq. (2.7)) it is easy to evaluate fixed-order perturbative coefficients. In Table 1, second to fourth column, we give values for the coefficients

n	$d_n^{\tau,\overline{\text{MS}}}$	$d_n^{\tau,V}$	$M_n^{\tau,\overline{\text{MS}}}$	$d_n^{D,\overline{\text{MS}}}$	$M_n^{\tau,LDP}$
0	1	1	1	1	1.329
1	2.2751	0.6084	1.521	0.6918	1.578
2	5.6848	0.8788	1.819	3.1035	1.855
3	13.754	-0.3395	1.984	2.1800	1.898
4	35.147	3.7796	2.081	30.740	2.027
5	84.407	-14.680	2.134	-34.534	2.000
6	248.83	99.483	2.170	759.74	2.094
7	525.38	-664.00	2.187	-3691.4	2.041
8	3036.0	5400.06	2.210	42251	2.042

Table 1: Coefficients for n fermion loop insertions into one-loop radiative corrections for the τ -lepton hadronic width and partial sums of the perturbation theory for $\alpha_s(m_\tau) = 0.32$ [$a_s(m_\tau) = 0.229$]. See text.

d_n defined in Eq. (2.3) for $n \leq 8$ in the $\overline{\text{MS}}$ - and V-scheme⁷ ($C = -5/3$ and $C = 0$), and partial sums of the perturbative series $M_n(a_s(m_\tau))$ in the $\overline{\text{MS}}$ -scheme, defined in Eq. (2.4), for $\alpha_s^{\overline{\text{MS}}}(m_\tau) = 0.32$ [22] and taking $N_f = 3$ massless flavours.

Truncating the expansion of R_τ at its minimal term, one deduces from the fourth column of Table 1 that the effect of summation of $(-\beta_0 \alpha_s(m_\tau))^n$ -corrections increases the leading order correction by factor $M_7^{\tau,\overline{\text{MS}}}(0.229) \simeq 2.19$, which is to be compared with 1.803, obtained from the exact coefficients up to order α_s^3 . The value of $M_7^{\tau,\overline{\text{MS}}}(a_s)$ from the truncated series is in good agreement with the result of resummation⁸,

$$M_\infty^{\tau,\overline{\text{MS}}}(0.229) = 2.233. \quad (3.11)$$

Taken at face value, higher order vacuum polarization effects lead to a significant increase of radiative corrections beyond the α_s^3 -approximation. The cumulative effect amounts to somewhat more than the α_s^3 -correction itself.

Before we discuss its impact on the determination of α_s , we note that it has become customary not to use a fixed order approximation for R_τ , but to employ the approach of

⁷ Note that both series are far from the expected asymptotic behaviour. There are two reasons for this: First, for the current correlation function, the formal large- n behaviour is dominated by ultraviolet renormalons, whose suppression in the $\overline{\text{MS}}$ scheme with $C = -5/3$ is significant in low orders. Therefore, one does not expect to see sign-alternating behaviour in low orders. Second, the contour integration rearranges coefficients and postpones the onset of the asymptotic regime, see below.

⁸We have obtained this value in three different ways: (1) Starting from Eq. (2.24) with the leading order corrections to hadronic τ decays with finite gluon mass collected in Appendix B; (2) Taking the principal value Borel integral of Eq. (3.10) and (3) Computing the principal value integral of Eq. (3.9) along a circle of radius m_τ^2 in the complex s -plane and then taking the contour integral along the circle according to Eq. (3.5). The intrinsic uncertainty due to IR renormalons is small, ± 0.02 , since the leading pole at $u = 2$ disappears in R_τ in the large- β_0 limit.

Le Diberder and Pich (LDP) [10], who resum exactly the effect of the running coupling along the circle in the complex plane, but not on $Q^2 d\Pi/dQ^2$ itself. This resummation is motivated by the observation that evolution along the circle generates a series of large higher order corrections, which is convergent, but only barely so at the actual value of $\alpha_s(m_\tau)$. The resummation of Le Diberder and Pich (restricted to one-loop running of the coupling) is automatically included in our resumation, since the running coupling is not expanded in $\alpha_s(m_\tau)$ in the derivation of Eq. (2.24). It is yet instructive to apply the procedure of [10] to fixed order approximations for $Q^2 d\Pi/dQ^2$. The successive approximations for R_τ are then given by

$$\begin{aligned} M_N^{\tau,LDP}(a_s(m_\tau)) &\equiv \sum_{n=0}^N d_n^D A_n(a_s(m_\tau)) a_s(m_\tau)^n \\ A_n(a_s(m_\tau)) &= \int_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8} \right) \frac{1}{[1 + a_s(m_\tau) \ln(-s/m_\tau^2)]^{n+1}}, \end{aligned} \quad (3.12)$$

where d_n^D are the expansion coefficients of $Q^2 d\Pi/dQ^2$, defined as in Eq. (2.3). $M_N^{\tau,LDP}(a_s(m_\tau))$ and $d_n^{D,\overline{\text{MS}}}$ are given in the last two columns of Table 1. From this we would conclude from truncation of the series that $M_\infty^{\tau,\overline{\text{MS}}}$ is about 2.05 or even less since the coefficient $d_2^{D,\overline{\text{MS}}} = 3.10$ is almost a factor three larger than the exact α_s^3 -coefficient 1.26 for $Q^2 d\Pi/dQ^2$, which indicates that keeping vacuum polarization alone is not a good approximation⁹ for $Q^2 d\Pi/dQ^2$ beyond order α_s^2 .

The difference between $M_\infty^{\tau,\overline{\text{MS}}}(0.229) = 2.233$ and 2.05 can be explained by the different weight of infrared and ultraviolet regions of internal momenta for R_τ as compared to $Q^2 d\Pi/dQ^2$. Since the A_n in Eq. (3.12) are positive numbers of order one, the behaviour of $M_N^{\tau,LDP}(a_s(m_\tau))$ as function of N is controlled by the series for $Q^2 d\Pi/dQ^2$. The point is that the series for this quantity becomes dominated by ultraviolet regions of integration much earlier than R_τ itself, for which the contour integral suppresses the ultraviolet renormalon singularities as well as their residues. For instance, the ratio of the contribution of the leading infrared renormalon pole and the leading ultraviolet renormalon pole to the coefficients of the expansion for R_τ at order $n+1$ is given by

$$e^{20/3} \frac{20}{9} \left(\frac{1}{3}\right)^n n \quad (3.13)$$

and the crossover to the asymptotically expected ultraviolet renormalon dominance takes place at $n \approx 8 - 9$. For $Q^2 d\Pi/dQ^2$ we obtain instead

$$e^5 \frac{2}{3} \left(\frac{1}{2}\right)^n \frac{1}{n} \quad (3.14)$$

with crossover at $n \approx 4 - 5$, which is indeed confirmed by the coefficients for $d_n^{D,\overline{\text{MS}}}$ given in Table 1. The problem with the onset of ultraviolet renormalon dominance is that then one is forced to construct the analytic continuation of the Borel transform

⁹The discrepancy is partially removed, when two-loop running is incorporated, see Sect. 5.2.

in order to overcome the formal $1/m_\tau^2$ -uncertainty, associated with the truncation of the series due to ultraviolet renormalons. Since for $\alpha_s = 0.32$ this onset of divergence is expected at $n \sim 1/a_s \sim 4 - 5$, we see that, in the $\overline{\text{MS}}$ scheme, the resummation of Le Diberder and Pich can not be continued beyond $n \approx 5$ without running into this $1/m_\tau^2$ -uncertainty. It is the suppression of ultraviolet regions of integration that allows a fixed-order approximation of R_τ itself to higher n than five, which explains the proximity of $M_7^{\tau, \overline{\text{MS}}}(0.229)$ from truncation to 2.233 in this case. Recall that after resummation, there is no uncertainty left due to ultraviolet renormalons, because the sign-alternating behaviour is summable.

Let us turn to the determination of α_s . We may write the (normalized) hadronic τ decay width as

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2) S_{EW} \left\{ 1 + \delta^{(0)} + \delta_{EW} + \delta_p \right\}, \quad (3.15)$$

where S_{EW} and δ_{EW} are electroweak corrections, $\delta^{(0)}$ is the perturbative QCD correction (with quark masses set to zero) and δ_p denotes power suppressed corrections in $1/m_\tau^2$, including quark mass and condensate terms. We have borrowed the values for S_{EW} , δ_{EW} and δ_p from [21, 22]. To a very good approximation we can neglect the variation of δ_p with α_s and we have evaluated it at $\alpha_s(m_\tau) = 0.32$. Then the experimental value $R_\tau = 3.56 \pm 0.03$ (quoted from [22]), obtained from an average of branching ratio and lifetime measurements translates into a constant experimental value¹⁰

$$\delta_{exp}^{(0)} = 0.183 \pm 0.010 \quad (3.16)$$

for the perturbative QCD corrections. In Fig. 3 we have plotted the prediction for $\delta^{(0)}$ as a function of $\alpha_s(m_\tau)$ after resummation of one-loop running effects (recall for R_τ the exact α_s^2 - and α_s^3 -coefficients are very well approximated by one-loop running) as compared to the α_s^2 - and α_s^3 -approximation as well as the α_s^3 -approximation, including the resummation of [10]. We conclude that resummation of one-loop running reduces the central value for $\alpha_s(m_\tau)$ by approximately 10% to

$$\alpha_s(m_\tau) = 0.29. \quad (3.17)$$

There is some difficulty in assigning an error to this value, which is related to the extent to which one trusts the restriction to one-loop running effects as a good estimate of higher order coefficients, not known exactly at present. In view of the fact that one-loop running overestimates the coefficient d_2 of $Q^2 d\Pi/dQ^2$, one might consider $M_\infty^{\tau, \overline{\text{MS}}}(a_s)$ as an upper estimate for the effect of higher order perturbative corrections and the quoted value for $\alpha_s(m_\tau)$ as a lower estimate for the central value. On the other hand, we shall see in Sect. 5.2, that a partial inclusion of two-loop running points towards even a lower value of $\alpha_s(m_\tau)$. It seems safe to conclude that higher order corrections add up constructively and a cumulative effect is most likely to shift

¹⁰The latest experimental data, which we have not taken into account, seem to indicate a larger value for R_τ . In this case, too, resummation reduces $\alpha_s(m_\tau)$ obtained without resummation by 10% – 15%.

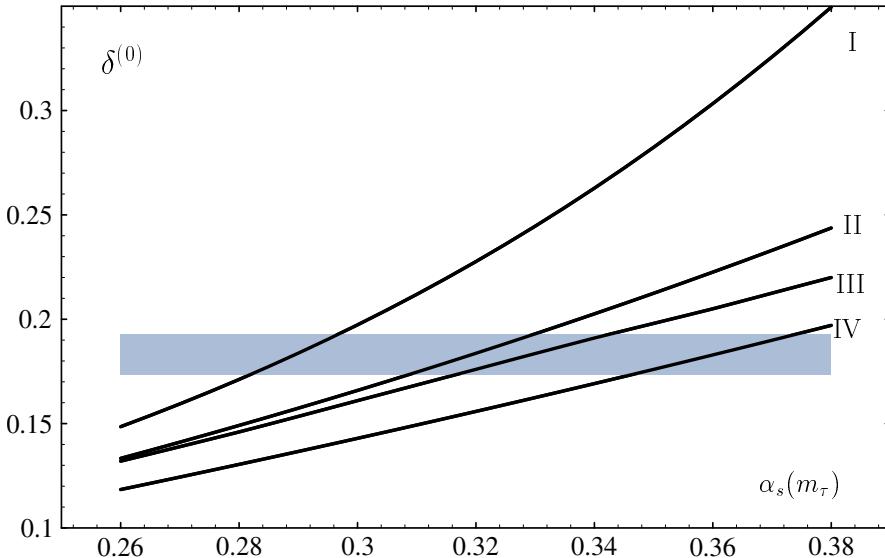


Figure 3: Perturbative corrections to the τ hadronic width. I: After resummation of one-loop running effects; II: Exact order α_s^3 -approximation; III: Exact order α_s^3 -approximation including the resummation of running coupling effects along a circle in the complex plane (taken from Pich); IV: Exact order α_s^2 -approximation for comparison. The shaded bar gives the experimental value with experimental errors only.

$\alpha_s(m_\tau)$ towards the above value. Optimistically, one could even hope for a reduction of the theoretical uncertainty from the unknown residual perturbative corrections. A conservative evaluation would not exclude the entire region from 0.27 to 0.34 for α_s at the scale m_τ . The lower bound reflects experimental and other than perturbative theoretical uncertainties and the upper bound follows from an analysis that takes into account only the completely known perturbative corrections up to cubic order.

The implications of the expected ultraviolet renormalon dominance for R_τ in large orders, which we have briefly touched upon above, have recently been investigated in Ref. [27], whose authors use conformal mapping techniques to construct the analytic continuation of the Borel transform beyond its radius of convergence set by the first ultraviolet renormalon. As noted in [27], such a technique is not particularly successful, when the series is not already close to the asymptotic ultraviolet renormalon behaviour (which is not the case for R_τ and $Q^2 d\Pi/dQ^2$ up to cubic order), because then the intricate cancellations necessary to push the ultraviolet renormalon further away from the origin of the (conformally mapped) Borel plane than the first IR renormalon do not take place. The variation of results obtained from different mapping functions can then be considered analogous to the variations induced by different choices of renormalization schemes and as in the latter case it is difficult to decide to what extent such a (in principle arbitrary) variation should be considered as a theoretical error. Based on the evidence presented above, we believe that ultraviolet renormalons are sufficiently

suppressed in fixed-order perturbative approximations to R_τ (but potentially not to $Q^2 d\Pi/dQ^2$) to be safely ignored at present.

3.2 $(\Lambda_{\text{QCD}}/m_\tau)^2$ -corrections

In this Subsection we investigate whether resummation of perturbative corrections can introduce power corrections, which elude the operator product expansion. We will exclude from the discussion the effect of renormalons, which have received main attention in this context [28, 11, 29, 14, 30, 27, 22]. As far as evidence from explicit calculations is available, infrared renormalons are in correspondence with condensates in the OPE as expected. There is no indication for explicit power corrections of dimension two from this source, which could turn out to be numerically significant. The effect of the dominant ultraviolet renormalon divergence is taken into account automatically by the error estimate due to unknown higher order perturbative corrections. After resummation it disappears completely in principle¹¹, although in practice this is not simple to implement [27] without approximations (like large β_0). It is also conceptually important to distinguish the statement that $1/Q^2$ -corrections are absent in the OPE of correlation functions at euclidian momenta from that that $1/m_\tau^2$ -corrections are absent in R_τ . Validity of the first needs not imply the second, though the second will hardly be true without the first.

3.2.1 Definition of the coupling

Our first point of concern is the definition of the coupling parameter inherent to perturbative expansions and therefore to their resummations. To emphasize that this question is not connected with the minkowskian nature of R_τ , we work with the derivative of the correlation function $D(Q^2)$ rather than with R_τ itself.

Let us first take a closer look at the Borel sum (to be definite, let us assume a principal value prescription throughout this subsection, which corresponds to taking the real part of $r_0(\lambda_L^2)$) in the representation as an integral over finite gluon mass in the large- β_0 (NNA) approximation. Let us denote the two terms on the right hand side of Eq. (2.24) by $I(\alpha_s)$ for the integral and $L(\alpha_s)$ for the contribution from the Landau pole in the dispersion relation for the running coupling (using the technique of Sect. 2.3 or [4]). The leading (two-loop) radiative correction with finite λ can be obtained from the Borel transform by a Mellin transformation, see Eq. (2.16). Taking the integral analytically can be difficult, but for our present purpose we are interested only in the coefficients in the small- λ^2 expansion. This expansion can be obtained easily, since particular terms proportional to $(\lambda^2/m_\tau^2)^n$ (modulo logarithms) correspond to contributions from singularities of the integrand in the right u -plane at $u = n$. For example, to pick up the term of order λ^2/m_τ^2 one can evaluate the integrand in Eq. (2.16) at $u = 1$, except for $\Gamma(-u)$. For the D -function defined in Eq. (3.8) we obtain

¹¹This is seen explicitly in the representation Eq. (2.24) for the Borel integral, which avoids construction of the analytic continuation of the Borel transform beyond its radius of convergence set by the first ultraviolet renormalon.

$$4\pi^2 D(Q^2, \lambda^2) = 1 + \frac{\alpha_s}{\pi} \left\{ 1 - \left[\frac{32}{3} - 8\zeta(3) \right] \frac{\lambda^2}{Q^2} - \left[2 \ln(Q^2/\lambda^2) + \frac{20}{3} - 8\zeta(3) \right] \frac{\lambda^4}{Q^4} + O(\lambda^6 \ln^2 \lambda^2/Q^2) \right\}. \quad (3.18)$$

The presence of quadratic terms, λ^2/Q^2 , is not in conflict with the operator product expansion, since such terms come from the region of large momenta: As emphasized in Sect. 2.4, only non-analytic terms in the small λ^2 -expansion can unambiguously be identified with infrared contributions. The leading non-analytic term is proportional to $\lambda^4 \ln \lambda^2$ and produces a correction proportional to $1/Q^4$, which can be related to the contribution of the gluon condensate [31] in the OPE. This contribution agrees with the calculation of the gluon condensate with finite gluon mass in [32], after the corresponding Wilson coefficient is extracted¹². (The difference in the constant 20/3 arises, because we consider the derivative of $\Pi(Q^2)$.)

The presence of a λ^2/Q^2 -correction in Eq. (3.18) implies that the Landau-pole contribution in Eq. (2.24) is

$$L(\alpha_s(Q)) \propto \exp\left(\frac{1}{\beta_0 \alpha_s(Q)}\right) \propto \frac{\Lambda_{\text{QCD}}^2}{Q^2}. \quad (3.19)$$

Numerically, this term is quite substantial. Taking $\alpha_s(Q) = 0.32$ (which corresponds to $Q \simeq m_\tau$), separation of the two terms in Eq. (2.24) for $D(Q^2)$ amounts to

$$M_\infty^D(0.229) = 1.48 = \text{integral over gluon mass} + 0.31. \quad (3.20)$$

Note that $L(\alpha_s)$ has identically vanishing perturbative expansion. Thus, without any additional information, keeping $I(\alpha_s)$ alone in Eq. (2.24) provides an equally legitimate summation of the original series, which differs from Borel summation by terms of order $1/Q^2$, which are not related to renormalons or any particular regime of small or large momenta. We conclude, at this stage, that statements about power corrections are meaningful only with respect to particular summation prescriptions.

Physically, dropping the contribution $L(\alpha_s)$ is equivalent to dropping the Landau pole contribution to the dispersion relation for the running coupling which can be interpreted as a redefinition of the coupling, such that the new coupling has no Landau pole. Since Borel summation in our limit of large β_0 coincides (in the V-scheme, $C=0$)

¹²The full contribution of order λ^4/Q^4 which is multiplied by $\ln(Q^2/\lambda^2) + const$ should be schematically decomposed as $\ln(\mu^2/\lambda^2)$, contributing to the gluon condensate, and $\ln(Q^2/\mu^2) + const$, contributing to the coefficient function to α_s accuracy. Here μ is the scale separating small and large distances. If one considers λ as an infrared cutoff, which is natural, then the full contribution should be ascribed to the coefficient function. Hence the rationale of ascribing the constant term to contributions of large momenta (small distances). In general, the separation of matrix elements and coefficient functions is of course factorization scheme-dependent and constant terms can be reshuffled. We see once more that only non-analytic terms (logarithmic in this example) can be used to trace infrared contributions.

with averaging the running coupling $\alpha_s(k) = \alpha_s(Q)/(1 - \beta_0 \alpha_s(Q) \ln(k^2/Q^2))$, it is readily seen from Eq. (2.33) that neglecting $L(\alpha_s)$ corresponds to averaging with a coupling $\alpha_s^{\text{eff}}(Q)$, related to $\alpha_s(Q)$ by

$$\alpha_s^{\text{eff}}(Q) = \alpha_s(Q) - \frac{\lambda_L^2}{\beta_0(Q^2 + \lambda_L^2)} = \alpha_s(Q) - \frac{1}{\beta_0} e^{1/(\beta_0 \alpha_s(Q))} + O(e^{2/(\beta_0 \alpha_s(Q))}). \quad (3.21)$$

This coupling has no Landau pole and freezes to a finite value as Q^2 approaches zero. However, it has $1/Q^2$ -corrections to its evolution and correspondingly to the β -function:

$$\beta^{\text{eff}}(\alpha_s^{\text{eff}}) = \beta_0(\alpha_s^{\text{eff}})^2 + \left(\frac{1}{\beta_0} + 2\alpha_s^{\text{eff}} \right) e^{1/(\beta_0 \alpha_s^{\text{eff}})} + O(e^{2/(\beta_0 \alpha_s^{\text{eff}})}) \quad (3.22)$$

Note that the absence of a Landau pole in α_s^{eff} is seen only after summation of an *infinite* number of power corrections in $1/Q^2$ (exponentially small terms in $\alpha_s(Q)$) in Eq. (3.21). Further, the coefficients of perturbative expansions of any quantity are the same whether one uses $\alpha_s(Q)$ or $\alpha_s^{\text{eff}}(Q)$ and diverge although the average

$$\int d^4k F(k, Q) \frac{\alpha_s^{\text{eff}}(k)}{k^2} \sim I(\alpha_s(Q)) \quad (3.23)$$

has no Landau pole ambiguities. We emphasize once more that averaging one-loop radiative corrections with this freezing coupling differs from Borel summation of the series in $\alpha_s(Q)$ (which gives $I(\alpha_s(Q)) + L(\alpha_s(Q))$) by $1/Q^2$ -terms. So does Borel summation of the identical series in $\alpha_s^{\text{eff}}(Q)$, because the couplings differ by such terms. We shall argue that couplings like $\alpha_s^{\text{eff}}(Q)$ obscure the relation to the operator product expansion in the sense that explicit $1/Q^2$ -terms must be added to the resummed result (such as $L(\alpha_s(Q))$), had one used $\alpha_s^{\text{eff}}(Q)$ in order to cancel spurious $1/Q^2$ -effects in the large Q^2 -expansion of the resummed result. This remark applies identically to a freezing coupling of type $1/(-\beta_0 \ln(c + Q^2/\Lambda^2))$ and thus to the procedure of [6], where such a coupling has been used to estimate the size of infrared contributions for quantities related to heavy quark expansions.

We will now try to make precise the statement that $1/Q^2$ -terms should be absent in the OPE. In order to talk about power corrections we have to attach some meaning to the divergent perturbative expansion at leading order in $1/Q^2$. A natural (though one might still ask whether it is justified to prefer Borel summation-type schemes to any other) definition apparently is

$$\left| D(Q^2)_{\text{exact}} - BS[D(\alpha_s(Q))_{\text{pert}}] \right| \sim O\left(\frac{1}{Q^4}\right), \quad (3.24)$$

where $BS[D(\alpha_s(Q))_{\text{pert}}]$ denotes the Borel integral (with principal value prescription) of the perturbative expansion of D in some $\alpha_s(Q)$. However, this statement is still ambiguous, because it implies knowledge of the Q^2 -dependence in the coupling, which is arbitrary to a large extent. Moreover, it is not sufficient to appeal to the usual ambiguities in the choice of perturbative renormalization schemes, since the coupling

and its evolution must be specified to power-like accuracy. We can bypass this point, noting that QCD with massless fermions has only a single free parameter. Since we are discussing asymptotic expansions in a dimensionful parameter $1/Q^2$, it seems most natural to choose a physical mass scale as this parameter, say m_ρ^2 . This is especially natural in the context of lattice definitions of QCD, where one would trade the bare coupling for m_ρ^2 . Then, if the operator product expansion exists nonperturbatively, this suggests the existence of a double expansion in $1/\ln(Q^2/m_\rho^2)$ and m_ρ^2/Q^2 at large Q^2 and the interpretation of the statement that there are no $1/Q^2$ terms could be¹³

$$\left| D(Q^2)_{\text{exact}} - BS[D(Q^2)_{\text{pert}}] \right| \leq K(\delta, R) \left(\frac{m_\rho^2}{cQ^2} \right)^2. \quad (3.25)$$

Here $BS[D(Q^2)_{\text{pert}}]$ denotes the Borel sum (with principal value prescription) of the leading term in the expansion of $D(Q^2)$ at large Q^2 , which is an infinite series in $1/\ln(cQ^2/m_\rho^2)$, where c is a constant to be specified later. The constant $K(\delta, R)$ depends on the opening angle δ and radius R of the sector in the complex $1/Q^2$ -plane, where Eq. (3.25) is supposed to hold. In general, $K(\delta, R)$ will neither be continuous nor bounded as a function of these two parameters. In particular, one can not expect a uniform bound in the entire cut plane. In the limit $\delta \rightarrow \pi$, this is related to violations of duality. We stress that as far as mathematical rigour is concerned, the validity of Eq. (3.25) must be regarded as purely hypothetical. We wish to present it as a mathematical formulation of the *assumption* that the operator product expansion holds at *euclidian* momenta (i.e. $\delta > 0$) and no $1/Q^2$ -terms are present in the asymptotic expansion. We note that the condition Eq. (3.25) is stronger than the condition that long-distance contributions can be factorized into condensates, which does not exclude the presence of power-like corrections, in particular $1/Q^2$, to coefficient functions, in particular of the unit operator, from short distances.

After we have chosen m_ρ as the fundamental parameter of QCD, we may define the coupling by its beta-function and an overall scale. We define (again, to leading- β_0 accuracy)

$$\alpha_s(Q) = \frac{1}{-\beta_0 \ln(cQ^2/m_\rho^2)} \equiv \frac{1}{-\beta_0 \ln(Q^2/\Lambda_V^2)}, \quad (3.26)$$

where we have fixed c by matching the large Q^2 behaviour with the V -scheme¹⁴. It is easy to see that Eq. (3.25) implies Eq. (3.24) with this coupling, which by definition

¹³ This is still a simplification. In fact, $\ln \ln(Q^2/m_\rho^2)$ is also expected to appear. We will restrict the discussion to the large- β_0 limit, which might be considered as an analytic continuation to a large negative number of massless fermion flavours. In this limit, only $\ln(Q^2/m_\rho^2)$ appears and Borel transformation with respect to $\ln(cQ^2/m_\rho^2)$ has a unique meaning.

¹⁴This is a matter of convenience, since it eliminates writing C in the large- β_0 approximation. We could also have matched to the perturbative $\overline{\text{MS}}$ coupling. We also note that the change of variables from m_ρ^2 to $\alpha_s(Q)$ is singular due to the Landau pole at Λ_V^2 . However, this is not a restriction, since Eq. (3.25) is limited to finite R anyway. The position of the pole may be varied by the choice of c , which implies a reorganization of powers in inverse logarithms, but the pole occurs always at a finite value of Q^2 .

has no power-like evolution. This implication is not valid for couplings, that incorporate $1/Q^2$ -dependence in their running. It is in this sense that we believe the use of freezing couplings is hazardous. If Eq. (3.25) is correct, then the use of Borel summation for perturbative series expressed in terms of such a coupling, or averaging lowest order radiative corrections with such a running coupling, necessitates the addition of explicit $1/Q^2$ -corrections simply to cancel such corrections hidden in the definition of the coupling.

In practice, in one way or another, one relates physical quantities and an unphysical coupling like Eq. (3.26) might be considered as an intermediate concept only. However, the importance of being definite with the evolution of the coupling to power-like accuracy is not diminished by the use of physical couplings. To give a somewhat constructed example: If one expressed $R_{e^+e^-}$ as an expansion in the effective coupling, defined by QCD corrections to the Gross-Llewellyn-Smith sum rule, one would expect $1/Q^2$ -corrections to this perturbative relation, which are imported from the definition of the coupling.

3.2.2 Analyticity and the Landau pole

After renormalization group improvement, perturbative expansions are plagued by the Landau pole. This unphysical singularity is endemic not only to perturbative expansions, but to the operator product expansion, truncated to any finite order. It requires some care in defining resummations for quantities like R_τ , which are related to euclidian quantities by analyticity. For the purpose of illustration, we shall restrict ourselves to the approximation, where $\beta(\alpha_s) = \beta_0 \alpha_s^2$ and adopt the V-scheme, $C = 0$, in all explicit formulas. Then, ignoring an irrelevant Q^2 -independent subtraction, we can write the Borel sum (with principal value prescription, as usual) as

$$\begin{aligned} BS[\Pi](Q^2) &\equiv \frac{1}{-\beta_0} \int_0^\infty du e^{-u/(-\beta_0 \alpha_s(\mu))} B[\Pi](u, Q^2/\mu^2) = \frac{1}{-\beta_0} \int_0^\infty du \left(\frac{\Lambda_V^2}{Q^2} \right)^u F(u) \\ &= BS[\Pi]_<(Q^2, u_0) + BS[\Pi]_>(Q^2, u_0), \end{aligned} \quad (3.27)$$

where we use $\alpha_s(Q) = 1/(-\beta_0 \ln(Q^2/\Lambda_V^2))$ and $F(u)$ is Q^2 -independent. In the second line, we have defined two new functions by splitting the Borel integral into two regions from 0 to u_0 and u_0 to ∞ . From the explicit form in Eq. (3.9), we deduce that, for *any* (positive) u_0 , $BS[\Pi]_<(Q^2, u_0)$ is analytic in the Q^2 -plane cut along the negative axis. On the other hand the u -integral in $BS[\Pi](Q^2)$ diverges, when $Q^2 < \Lambda_V^2$ and $Q^2 = \Lambda_V^2$ is a singular point. We say that $BS[\Pi](Q^2)$ has a Landau pole, though, in general, $Q^2 = \Lambda_V^2$ will rather be a branch point.

Notice, since in practice one truncates the operator product expansion at operators of some dimension d , it is perfectly consistent to replace $BS[\Pi](Q^2)$ by $BS[\Pi]_<(Q^2, u_0)$, provided we choose $u_0 > d/2 + 1$, since the difference is bounded by

$$|BS[\Pi](Q^2) - BS[\Pi]_<(Q^2, u_0)| < \tilde{K}(R) \left(\frac{\Lambda_V^2}{Q^2} \right)^{u_0} \quad (3.28)$$

and vanishes faster than other corrections neglected in the truncation of the OPE. In this case, the bound can be established in a cut circle of radius R in the $1/Q^2$ -plane. Again, it will not be possible to establish uniform bounds, since typically $\tilde{K}(R) \sim 1/\ln(R\Lambda_V^2)$. Thus, although for fixed $Q^2 > 1/R$, the difference can be made vanish faster than any desired power of Λ_V^2/Q^2 by increasing u_0 , at fixed u_0 the bound may become arbitrarily weak as Q^2 approaches Λ_V^2 . Still, we conclude that the presence or absence of a Landau pole in resummed results is related to the behaviour of the Borel integral at infinity and is thus an effect that formally vanishes faster than any power of $1/Q^2$.

Consider now the two different representations for the tau decay width, Eqs. (3.3) and (3.5), in this light. We use equality of vector and axial-vector correlators in perturbation theory and abbreviate the weight function by $w(s/m_\tau^2)$. Then

$$R_\tau^{\text{eq.(3.3)}} = 24\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} w\left(\frac{s}{m_\tau^2}\right) \frac{1}{2i} \text{disc } BS[\Pi](s), \quad (3.29)$$

$$R_\tau^{\text{eq.(3.5)}} = 12\pi i \int_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} w\left(\frac{s}{m_\tau^2}\right) BS[\Pi](s). \quad (3.30)$$

Note that in both cases summation is carried out inside the s -integral. The following considerations can easily be extended to the situation, where summation is taken after s -integration (and do not lead to any of the differences observed below). We define $R_{\tau,<}^{\text{eq.(3.3)}}(u_0)$ and $R_{\tau,<}^{\text{eq.(3.5)}}(u_0)$ by the replacement of $BS[\Pi](s)$ by $BS[\Pi]_<(s, u_0)$. Using the analyticity properties discussed above as well as $m_\tau^2 > \Lambda_V^2$, it is straightforward to find

$$R_\tau^{\text{eq.(3.5)}} - R_{\tau,<}^{\text{eq.(3.5)}}(u_0) \sim \left(\frac{\Lambda_V^2}{m_\tau^2}\right)^{u_0} \quad R_{\tau,<}^{\text{eq.(3.3)}}(u_0) = R_{\tau,<}^{\text{eq.(3.5)}}(u_0), \quad (3.31)$$

but

$$R_\tau^{\text{eq.(3.3)}} - R_\tau^{\text{eq.(3.5)}} = -12\pi i \int_C \frac{ds}{m_\tau^2} w\left(\frac{s}{m_\tau^2}\right) BS[\Pi]_>(s, u_0), \quad (3.32)$$

where the contour C runs along a circle of radius Λ_V^2 around $s = -\Lambda_V^2$. Against appearances, the right hand side is independent of u_0 . Of course, one can not take u_0 to infinity inside the integral and conclude that it is zero. $BS[\Pi]_>(s, u_0)$ has a pole or branch point at $s = -\Lambda_V^2$ and we find

$$R_\tau^{\text{eq.(3.3)}} - R_\tau^{\text{eq.(3.5)}} \sim \frac{\Lambda_V^2}{m_\tau^2} \quad (3.33)$$

and similarly

$$R_\tau^{\text{eq.(3.3)}} - R_{\tau,<}^{\text{eq.(3.3)}}(u_0) \sim \frac{\Lambda_V^2}{m_\tau^2}. \quad (3.34)$$

Since Eq. (3.28) is not valid for $s < \Lambda_V^2$, this result should not surprise. The difference in Eq. (3.33) arises, because the resummation introduces (or preserves) the Landau pole singularity, which is in conflict with the analytic properties of the exact correlation function, that have been assumed to derive Eq. (3.5) from Eq. (3.3).

Should one conclude then that resummations of perturbative corrections are ambiguous by terms of order $1/m_\tau^2$, even if there are no $1/Q^2$ terms in the OPE at euclidian momenta in the strong sense of Eq. (3.25)? Do we have evidence for power corrections not captured by the OPE, since they originate from $u = \infty$ in the Borel integral? The answer is no. A positive answer would be warranted, if there were no reason to prefer the prescription Eq. (3.29) to Eq. (3.30) or vice versa, while only the second can be used. The reason is that the region $|s| < \Lambda_V^2$ can not be penetrated to any finite order in the short distance expansion ($1/Q^2$ -expansion). Since all summation prescriptions of perturbative expansions are formulated within the context of the short-distance expansion¹⁵, they can not be applied to $|s| < \Lambda_V^2$, unless the summation of this expansion itself is understood. This discards Eq. (3.29) as a legitimate summation. One must first use the analyticity properties of the exact correlation functions to deform the contour outside the region $|s| < \Lambda_V^2$, before an attempt at summing perturbative expansions can be made, which privileges Eq. (3.5) as the starting point. A different way to express this fact is to observe that the principal value Borel integral is defined for $|s| < \Lambda_V^2$ only in the sense of analytic continuation, but cannot be used as a numerical approximation since all power corrections in the operator product expansion are of the same order of magnitude in this region. It is only when all these are taken into account that the Landau pole vanishes in physical observables.

To conclude this Section, let us mention that a relation between analyticity in s and behaviour of the Borel integral at infinity has been noted in a very different and much more physical context in [33]. The presence of resonances and multiple thresholds on the physical axis was observed to be in conflict with Borel summation¹⁶. In this case, the restoration of the correct analytic properties should also be understood in connection with summing the OPE itself [35]. A much more elaborate argument has been presented in [36], where the presence of resonances was connected with the divergence of OPE, which also offers a way to understand the concept of duality and its limitations. If the OPE is by itself only an asymptotic expansion, it is quite possible that its application is limited to a finite phase range in the complex s -plane around the negative s -axis. We believe that this possibility should be taken seriously, since it might imply the presence of $1/s_0$ -corrections in finite averages up to s_0 of the discontinuity along the physical axis (for R_τ , $s_0 = m_\tau^2$). Unfortunately, we do not know how to substantiate or disprove such a statement theoretically.

¹⁵ See the discussion below Eq. (2.8).

¹⁶ Historically, it is interesting to note that the horn-shaped analyticity region in the coupling, which leads to this conclusion, was discovered for the photon propagator in [34].

4 The pole mass of a heavy quark

Although on-shell quarks do not exist and there is presumably no natural nonperturbative definition of a pole mass of a quark, the pole mass has proven useful as an auxiliary concept in applications of perturbative QCD to heavy quark physics, where physically the quark is expected to be close to its would-be mass-shell. Still, the relation between the pole mass and an off-shell renormalized mass is known to have large perturbative coefficients from small loop momenta, at least in high orders of perturbation theory [17, 37]. Since neither quark mass definition is physical, this might appear as an irrelevant problem. However, the behaviour of perturbative expansions of quantities involving quark masses change accordingly with the quark mass definition used and in general one expects coefficients to be significantly reduced, when the pole mass is abandoned in favour of an off-shell renormalized mass [37, 20], such as $\overline{\text{MS}}$. In the latter case, it is quite well-known that the exact two-loop coefficient [38] in the relation to the pole mass is substantial and one might wonder whether this is coincidental.

In this Section we apply resummation and NNA to the difference between the pole and $\overline{\text{MS}}$ mass. Apart from its practical interest, we also use this quantity to illustrate certain features of higher order perturbative corrections, when the masses of fermions in loops are finite. The reader interested only in results may jump directly to Sect. 4.3, where our best estimate is presented. In this Section $\alpha_s(\mu)$ always refers to the $\overline{\text{MS}}$ coupling.

4.1 Preliminaries

To begin with, let us quote the exact two-loop result from [38]:

$$\frac{\delta m}{m} \equiv \frac{m_{\text{pole}} - m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})}{m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})} = \frac{4}{3} \frac{\alpha_s(m_{\overline{\text{MS}}})}{\pi} \left[1 + \left(4.68 (-\beta_0^{N_f}) - \begin{Bmatrix} 0.89 \\ 0.91 \end{Bmatrix} \right) \alpha_s(m_{\overline{\text{MS}}}) \right] \quad (4.1)$$

The upper entry in curly brackets refers to the calculation with N_f massless quarks and a quark of mass m in loops, the lower entry to the situation, where the quark flavour of mass m is excluded from loops¹⁷. The superscript on β_0 indicates the value of N_f to be taken for the β -function. Since $(-\beta_0^{N_f}) \sim 2/3$ for cases of interest, $N_f = 3, 4$, we note that keeping the term proportional to $\beta_0^{N_f}$ alone provides indeed a reasonable approximation within 30 – 40% of the exact coefficient.

Note that above we have normalized $m_{\overline{\text{MS}}}$ at the scale $m_{\overline{\text{MS}}}$, since we prefer to eliminate m_{pole} in all places. When Eq. (4.1) is expressed in terms of the $\overline{\text{MS}}$ mass normalized at m_{pole} ,

$$\frac{m_{\text{pole}} - m_{\overline{\text{MS}}}(m_{\text{pole}})}{m_{\overline{\text{MS}}}(m_{\text{pole}})} = \frac{4}{3} \frac{\alpha_s(m_{\text{pole}})}{\pi} \left[1 + \left(4.68 (-\beta_0^{N_f}) - \begin{Bmatrix} 0.25 \\ 0.28 \end{Bmatrix} \right) \alpha_s(m_{\text{pole}}) \right] \quad (4.2)$$

¹⁷To obtain the lower value, it is necessary to insert a missing C_F in front of the mass correction in Eq. (17) of [38], see also [3].

the approximation of keeping only the term proportional to $\beta_0^{N_f}$ is significantly improved. This ambiguity is in fact a source of trouble within the BLM and our extended prescription. For δm additional renormalization scheme and scale dependence is present from the definition of the quark mass parameter, whereas the BLM method by construction deals only with the scale-dependence of the coupling. Although the calculation of fermion loop insertions and subsequent restoration of β_0 also provides the anomalous dimension of the quark mass within the same approximation, the scheme ambiguity in the size of neglected genuine two-loop corrections is amplified by this additional source of scheme-dependence. If $\mu_1^2 - \mu_2^2 \sim O(\alpha_s)$, then the difference $m_{\overline{\text{MS}}}(\mu_1) - m_{\overline{\text{MS}}}(\mu_2)$ must be neglected in the approximation of large β_0 , although it can be sizeable, if the coefficient of α_s in $\mu_1^2 - \mu_2^2$ is so. In the following, we will work with Eq. (4.1), whenever relevant.

Let us first consider the (excellent) approximation, that the quark flavour of mass m is neglected in loops. All other N_f quarks are massless. In this case the relevant β -function coefficient is $\beta_0^{N_f}$ and an exact expression for the Borel transform of the mass shift exists [17]:

$$B[\delta m/m](u) = \frac{1}{3\pi} \left[e^{5u/3} 6(1-u) \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} + \frac{\tilde{G}_0(u)}{u} \right] \quad (4.3)$$

$\tilde{G}_0(u)$ is defined in the following way (see Appendix A): If g_n are the expansion coefficients of $G_0(u)$ in u , then $\tilde{G}_0(u)$ has expansion coefficients $g_n/n!$. $G_0(u)$ can be calculated with the methods of Appendix A and is found to be

$$G_0(u) = -\frac{1}{3}(3+2u) \frac{\Gamma(4+2u)}{\Gamma(1-u)\Gamma^2(2+u)\Gamma(3+u)}. \quad (4.4)$$

Higher order perturbative corrections with fermion loop insertions into the one-loop diagram can then be computed according to Eq. (2.7). Equivalently, one can start from Eq. (2.28) with the input of the one-loop mass shift with finite gluon mass, given by ($x \equiv \lambda^2/m^2$)

$$\begin{aligned} r_0(\lambda^2) &= \frac{1}{3\pi} \left[4 + x - \frac{x^2}{2} \ln x - \frac{\sqrt{x}(8+2x-x^2)}{\sqrt{4-x}} \left\{ \arctan \left[\frac{2-x}{\sqrt{x(4-x)}} \right] + \right. \right. \\ &\quad \left. \left. \arctan \left[\frac{\sqrt{x}}{\sqrt{4-x}} \right] \right\} \right] = \frac{4}{3\pi} \left[1 + \frac{\pi}{2}\sqrt{x} + \frac{3}{4}x + O(x^{3/2}) \right]. \end{aligned} \quad (4.5)$$

We define coefficients d_n as in Sect. 2 by

$$\frac{\delta m}{m} = \frac{4}{3} \frac{\alpha_s(m)}{\pi} \left[1 + \sum_{n=1}^{\infty} d_n (-\beta_0^{N_f} \alpha_s(m))^n \right]. \quad (4.6)$$

They are listed in Table 2. Higher order perturbative corrections grow very rapidly, as expected from the dominant asymptotic behaviour from the pole at $u = 1/2$ in Eq. (4.3),

n	d_n	M_n [c-quark]	M_n [b-quark]
0	1	1	1
1	4.6861511	2.176	1.623
2	17.622650	3.286	1.935
3	109.85885	5.024	2.193
4	873.92393	8.492	2.467
5	8839.6860	17.30	2.835
6	105814.28	43.76	3.420
7	1484968.4	137.0	4.514
8	237407365	511.0	6.838
∞	—	1.712 ± 0.608	2.041 ± 0.201
m_1^*	—	$0.096 m_c$	$0.096 m_b$
m_∞^*	—	$0.437 m_c$	$0.147 m_b$

Table 2: Higher order perturbative corrections to the difference between the pole and $\overline{\text{MS}}$ mass from n fermion loop insertions. The second and third columns give partial sums and summations for the charm quark ($a_s = -\beta_0^{(3)}\alpha_s(m_c) = 0.251$) and bottom quark ($a_s = -\beta_0^{(4)}\alpha_s(m_b) = 0.133$). In the latter case, the c-quark is treated as massless inside loops.

$$d_n \stackrel{n \geq 1}{\approx} e^{5/6} 2^n n!. \quad (4.7)$$

This asymptotic formula is in fact a very good approximation (within 5%) to the exact coefficient d_n for all $n \geq 1$, which seems to imply that the saddle point approximation of the loop momentum distribution inherent to deriving the asymptotic behaviour is a very good substitute for the exact distribution, even if the width of the Gaussian is not small. Given that genuine two-loop corrections are rather small compared to d_1 , it can be asserted that the exact two-loop coefficient is already dominated by the first infrared renormalon at $u = 1/2$. Evidently, this observation is scheme-dependent and we do not know why the $\overline{\text{MS}}$ scheme is preferred. Let us note, however, that a similar coincidence can not be expected and indeed does not happen (see Sect. 3) for quantities, whose asymptotic behaviour is dominated by ultraviolet renormalons, like $R_{e^+e^-}$ or R_τ , since with $C = -5/3$ in the $\overline{\text{MS}}$ scheme, the leading ultraviolet singularity at $u = -1$ is suppressed compared to the leading infrared singularity at $u = 2$ by a factor $e^{-5} \approx 7 \cdot 10^{-3}$. Therefore the latter is expected to dominate in intermediate orders [35] (if there is any regularity at all) and it might not be an accident, that exact low order coefficients are indeed of same sign for these quantities.

Table 2 also shows how the one-loop radiative correction to the pole mass is modified by inclusion of a finite number of fermion loops and by summation according to Eq. (2.29). We have taken $\alpha_s(m_b) = 0.2$ and $\alpha_s(m_c) = 0.35$. Recall that the sum

corresponds to a principal value prescription for the Borel integral. The errors quoted correspond to the imaginary part of the Borel integral, when it is defined by deforming the contour into the complex plane. The imaginary part can be taken as an estimate of inherent uncertainty of perturbative relations. We have actually divided this imaginary part by π , which upon inspection we find closer to, but somewhat smaller than the naive estimate of uncertainty by the minimal term of the series. A more conservative estimate would be to enlarge these errors by a factor of two. The BLM scales in Table 2 are defined by (cf. Eq.(1.5))

$$m_1^* = m \exp \left[-\frac{1}{2a_s} (M_1(a_s) - 1) \right], \quad m_\infty^* = m \exp \left[-\frac{1}{2a_s} \left(1 - \frac{1}{M(a_s)} \right) \right]. \quad (4.8)$$

The uncertainty in $M_\infty(a_s)$ translates into an uncertainty of the BLM scale m_∞^* by the previous equation. We want to point out that the usual BLM scale m_1^* , which uses only d_1 as input is smaller than m_∞^* although all higher orders d_n add up positively. The reason is that $\alpha_s(m_1^*)$ upon re-expanding in terms of $\alpha_s(m)$ implies $d_n^{\text{BLM}} = d_1^n$. For the charm and bottom quark mass the most important effect comes from $n = 1, 2$. But since d_1 is rather large, $d_2^{\text{BLM}} = d_1^2 \approx 21.9 > 17.6 = d_2$, and the usual BLM prescription overestimates the size of radiative corrections. This behaviour is quite generic to quantities dominated by scales below a few GeV and with a leading infrared renormalon at $u = 1/2$.

4.2 Effect of internal quark masses

Before proceeding to realistic charm and bottom quarks we want to illustrate the effect of finite quark masses inside loops on the coefficients of diagrams with quark loop insertions in the toy example of the mass shift due to a single quark flavour with mass m_i in loops. The “heavy” quark of mass m is again excluded from loops.

The factorially large contribution from small and large loop momenta can be traced to the logarithmic behaviour of the vacuum polarization for a massless particle at very small and very large virtuality. In the case of small virtuality, the large coefficient arises from momenta $k \sim me^{-n} \ll m$. For a massive quark the logarithmic behaviour is cut off, because at very small momenta its vacuum polarization is proportional to k^2/m_i^2 . Thus, no matter how small the quark mass, the factorially large contribution should be eliminated when n is such that $me^{-n} < m_i$. There are no infrared renormalon singularities in the Borel transform in the absence of massless particles. This expectation is verified in Fig. 4, where we have plotted the ratio of the coefficient $d_n(m_i)$, computed with internal mass m_i , and $d_n(0)$ for the massless case as in Table 2 as a function of the number of loops n for various ratios of m_i/m . Since m_i serves effectively as an infrared cutoff, Fig. 4 provides some information on what proportion of the coefficient $d_n(0)$ originates from low momentum regions. Note that low momentum means low momentum compared to m and not Λ_{QCD} . Eventually, as n becomes large, coefficients will be dominated by large momentum for any m_i and the asymptotic behaviour of Eq. (4.7) is replaced by a sign-alternating behaviour due the leading ultraviolet renormalon

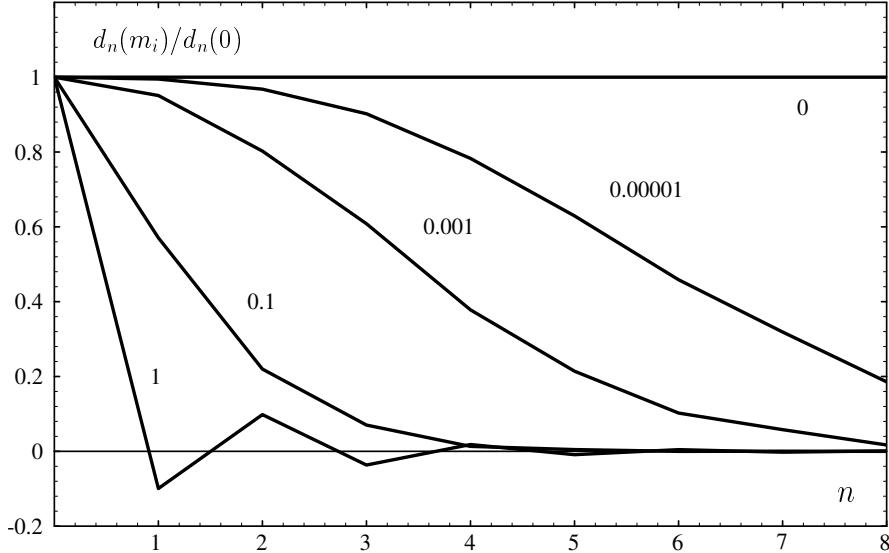


Figure 4: Ratio $d_n(m_i)/d_n(0)$ for different values of internal quark masses m_i as a function of the number of fermion loop insertions. The value of m_i^2/m^2 is indicated to the right of each curve.

$$d_n(m_i) \stackrel{n \gg 1}{\approx} \left[-1 + \frac{9}{2} \frac{m_i^2}{m^2} \right] e^{-5/3} (-1)^n n!. \quad (4.9)$$

We find that for $m_i^2/m^2 \geq 0.1$, this asymptotic behaviour practically coincides with the exact coefficient for $n > 6$.

The presence of mass dependence in the coefficient of ultraviolet renormalons requires some explanation. The singularity of the Borel transform at $u = -1$, which is responsible for Eq. (4.9), is due to the presence of a $1/k^6$ -term in the expansion of the Feynman integrand for the one-loop mass-shift at large k :

$$\delta m \sim \alpha_s m \int d^4 k \left(\frac{a}{k^4} + b \frac{m^2}{k^6} + \dots \right) \quad (4.10)$$

The $1/k^4$ -term creates a logarithmic ultraviolet divergence, which is subtracted by minimal subtraction. The effect of the Borel transformed gluon propagator (with fermion loop insertions) amounts to insertion of (see Sect. 2.3)

$$\exp(-u\Pi(k^2)/a_s) = \left(-\frac{m^2}{k^2} e^{-C} \right)^u \left[1 + 6u \frac{m_i^2}{k^2} + O\left(\frac{m_i^4}{k^4}\right) \right] \quad (4.11)$$

into the integrand. The pole at $u = -1$ arises, because close to $u = -1$ the $1/k^6$ -term is converted into a logarithmically divergent term. But when m_i is not zero a second contribution of order $1/k^6$ arises, when a/k^4 combines with the first mass correction

in Eq. (4.11). This term accounts for the mass-dependence of ultraviolet renormalons in minimal subtraction schemes. Had one chosen a renormalization prescription, which subtracts the a/k^4 -term inside the integrand, that mass-dependence would be absent for the first ultraviolet renormalon at $u = -1$, but still present in coefficients of singularities at $u = -2, -3 \dots$.

4.3 Charm and bottom pole masses

For realistic charm and bottom quarks, the numerical results of Sect. 4.1 need to be amended in two respects: The quark, whose mass shift is considered, should also be taken into account in loops. This effect is tiny, but we include it for completeness. More interesting is the effect of finite charm mass on the bottom pole mass, because the charm mass might be larger than the typical loop momentum already in low orders. If so, it would seem more appropriate to use $\beta_0^{(3)}$ rather than $\beta_0^{(4)}$, when restoring the QCD β -function coefficient from the calculation of fermion loops. Strictly speaking, changing the mass of one flavour is a negligible effect in the formal large- β_0 limit (i.e. large- N_f), and should be discarded in a consistent approximation. *In realiter* the effect is numerically noticeable and we prefer to include it as a calculable correction, keeping in mind its size as an uncertainty in our approach.

To gain some understanding of the numerical results to follow, we trace the decoupling of internal charm loops for the bottom pole mass in more detail. For this purpose, we ignore again the bottom quark in loops. When the order of perturbation theory is sufficiently large, internal integrations are dominated by momenta much smaller than m_c . Let us denote coefficients including charm loops by $d_n^{[3+1]}(m_c) (-\beta_0^{(4)})^n$ and those without charm loops by $d_n^{[3]} (-\beta_0^{(3)})^n$. Then one might expect

$$d_n^{[3+1]}(m_c) \left(-\beta_0^{(4)}\right)^n \alpha_s(m_b)^{n+1} \approx d_n^{[3]} \left(-\beta_0^{(3)}\right)^n \alpha_s(m_b)^{n+1} \quad (4.12)$$

if n is sufficiently large up to corrections suppressed by a power of the typical internal momentum divided by m_c . This is not quite correct, because there is no manifest decoupling in the $\overline{\text{MS}}$ scheme. In the limit that we consider only the contribution from small λ^2 , appropriate for sufficiently large n , the Borel transform including the massive charm in Eq. (2.37) reduces to

$$B \left[\frac{\delta m}{m} \right] (u) = \exp \left(\frac{u}{6\pi\beta_0^{(4)}} \ln \frac{\mu^2}{m_c^2} \right) \left(-\frac{1}{\pi} \right) \sin \left(\frac{u\beta_0^{(3)}}{\beta_0^{(4)}} \right) \int_0^\infty \frac{d\lambda^2}{\lambda^2} (r_0(\lambda^2) - r_0(0)) \left(\frac{\lambda^2}{\mu^2 e^{5/3}} \right)^{u\beta_0^{(3)}/\beta_0^{(4)}} \quad (4.13)$$

up to power corrections. If the first factor were absent, this Borel transform would coincide with the one for purely massless flavours up to a rescaling of u which is just what is necessary to obtain Eq. (4.12) from Eq. (2.7). The first factor is present, because the vacuum polarization for charm has been renormalized in the $\overline{\text{MS}}$ scheme and not by zero momentum subtraction, which would lead to manifest decoupling. The difference can be accommodated by a change of the coupling constant. This is most easily seen

n	$d_n^{[3]}$	$d_n^{[3+1]}$	$d_n^{[3]} (\beta_0^{(3)} / \beta_0^{(4)})^n$	$M_n^{[3+1]}$
0	1	1	1	1
1	4.6862	5.0984	5.0613	2.183
2	17.623	20.514	20.555	3.291
3	109.86	138.58	138.38	5.021
4	873.92	1188.3	1189.0	8.470
∞	—	—	—	1.720 ± 0.608
m_1^*	—	—	—	$0.078 m_c$
m_∞^*	—	—	—	$0.406 m_c$

Table 3: Perturbative corrections to the pole charm quark mass with (column 3) and without (column 2) charm inside loops. For comparison, column 4 gives coefficients in the limit that all internal masses are large compared to the typical loop momentum. The last column updates the modification of one-loop corrections due to coefficients $d_n^{[3+1]}$.

by combining the extra exponential factor with the exponential in the Borel integral, Eq. (2.8). We deduce that (up to power corrections in the typical loop momentum divided by m_c)

$$d_n^{[3+1]}(m_c) \left(-\beta_0^{(4)}\right)^n \alpha_s^{(4)}(m_b)^{n+1} \approx d_n^{[3]} \left(-\beta_0^{(3)}\right)^n \alpha_s^{(3)}(m_b)^{n+1}, \quad (4.14)$$

where the two couplings are related by

$$\frac{1}{\alpha_s^{(3)}(\mu)} = \frac{1}{\alpha_s^{(4)}(\mu)} + \frac{1}{6\pi} \ln \frac{\mu^2}{m_c^2}. \quad (4.15)$$

Alternatively, we can simply replace u in the extra exponential factor in Eq. (4.13) by the location of the closest infrared pole of the Borel transform, $u = 1/2$, and obtain

$$d_n^{[3+1]}(m_c) \left(-\beta_0^{(4)}\right)^n \alpha_s(m_b)^{n+1} \approx \exp\left(\frac{1}{12\pi\beta_0^{(4)}} \ln \frac{m_b^2}{m_c^2}\right) d_n^{[3]} \left(-\beta_0^{(3)}\right)^n \alpha_s(m_b)^{n+1}, \quad (4.16)$$

which is equivalent to Eq. (4.14) for large n .

Numerical values for corrections to the charm pole mass and the bottom pole mass, including the charm and bottom masses in loops, are given in Tables 3 and 4. For the bottom pole mass, the result depends only on the ratio m_c/m_b and we used $m_c^2/m_b^2 = 0.1$. For comparison coefficients according to Eq. (4.16) are given. The last column in both tables displays the modification of the one-loop mass shift by running coupling effects. Note that at the charm scale, the perturbative series has to be truncated already at second order. The lower rows give the result of summation according to Eq. (2.37) and the corresponding BLM scales, Eq. (4.8). Two technical observations are in order:

n	$d_n^{[3]}$	$d_n^{[3+2]}$	$K d_n^{[3]} (\beta_0^{(3)} / \beta_0^{(5)})^n$	$M_n^{[3+2]}$
0	1	1	1	1
1	4.6862	5.3093	4.9774	1.648
2	17.623	22.720	21.975	1.986
3	109.86	159.69	160.82	2.276
4	873.92	$1.5 \cdot 10^3$	1502.0	2.601
∞	—	—	—	2.099 ± 0.224
m_1^*	—	—	—	$0.064 m_b$
m_∞^*	—	—	—	$0.117 m_b$

Table 4: Coefficients as in the previous table, where $d_n^{[3+2]}$ now includes charm and bottom masses with $m_c^2/m_b^2 = 0.1$. $K = \exp(1/(12\pi\beta_0^{(5)}) \ln(m_b^2/m_c^2)) \approx 0.9$.

First, as mentioned in Section 2, the sum as defined by Eq. (2.37) does not exactly coincide with the principal value of the Borel integral. The difference is tiny for the value of masses considered here and does not affect our conclusions which are based on the behaviour of perturbation theory in low and intermediate orders. Second, for the bottom quark the BLM scale is defined without taking into account the charm threshold in the running coupling.

The difference between the pole mass and the $\overline{\text{MS}}$ -renormalized mass (normalized at $m_{\overline{\text{MS}}}$) for bottom quarks can now be written as

$$\frac{\delta m_b}{m_b} = \frac{4}{3} \frac{\alpha_s(m_b)}{\pi} \left[M_\infty^{[3+2]}(-\beta_0^{(5)} \alpha_s(m_b)) - 0.91 \alpha_s(m_b) + \text{higher orders} \right], \quad (4.17)$$

where the second term in square brackets accounts for the genuine gluonic two-loop corrections, cf. Eq. (4.1). Unknown higher order corrections include genuine three- and higher loop corrections, vacuum polarization insertions into two-loop corrections as well as effects of two-loop running on lowest order radiative corrections. Numerically, for $\alpha_s(m_b) = 0.2$, we find

$$\frac{\delta m_b}{m_b} = (16.3 \pm 2.9 \pm 1.5)\%. \quad (4.18)$$

Compared to the two-loop expression, the estimate obtained from NNA increases the mass shift from about 12% to 16%. The second error reflects our estimate of unknown higher order corrections, which we allow to be as large as the genuine two-loop corrections. The first error, which dominates the total uncertainty, represents an estimate of the ultimate accuracy of $\delta m_b/m_b$ due to the divergence of the perturbative series. It can not be reduced by calculating higher orders (but of course refined – the numerical value we quote has been obtained by increasing the uncertainty of $M_\infty^{[3+2]}(a_s)$ in Table 4 by 50%, upon which it is close to the minimal term of the series at $n \approx 2 - 3$). In

absolute values, the intrinsic uncertainty of δm_b and therefore of the pole mass of the bottom quark ranges between 100 and 150 MeV. The above errors do not include an uncertainty in $\alpha_s(m_b)$, which has been fixed to 0.2.

All results were given in terms of $m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})$. To run to a different renormalization point μ , one should use the anomalous dimension to the same large- β_0 approximation. In the $\overline{\text{MS}}$ -scheme it is given by

$$\gamma_m(\alpha_s) \equiv -\frac{\mu^2}{m} \frac{dm(\mu)}{d\mu^2} = -\frac{\alpha_s}{3\pi} G_0(-\beta_0 \alpha_s) \quad (4.19)$$

with G_0 as in Eq. (4.4) in agreement with [39]. Thus, in this approximation

$$m(\mu) = m(m) \exp \left(\int_{\alpha_s(m)}^{\alpha_s(\mu)} \frac{d\alpha_s}{3\pi\beta_0\alpha_s} G_0(-\beta_0 \alpha_s) \right). \quad (4.20)$$

To the approximation considered, one may replace the exponential by the first two terms of its expansion. In practice, it might be better to consider the approximation as an approximation to the exponent and to keep the exponential.

5 Scale-setting at next-to-leading order

So far we have been discussing a scale-setting procedure that utilizes only one-loop evolution of the strong coupling. At the same number of loops, where this procedure extends the familiar scheme of Brodsky, Lepage and Mackenzie, one also encounters diagrams associated with two-loop evolution, which are not suppressed compared to insertions of single fermion loops by any small parameter. This Section is devoted to the possibility of and difficulty in extending the scale-setting to next-to-leading order (NLO). The precise meaning of NLO in this context is illustrated in Fig. 5: For any process, we will calculate the class of higher order corrections, generated by substituting the gluon (photon) propagator in lowest order by the chains of Fig. 5a (this was done in previous Sections) and Fig. 5b. In the abelian theory, the results are exact, while in the nonabelian theory, one must again define how to restore the coefficients β_0 and β_1 of the nonabelian β -function. Let us note that as in the case of one-loop running this extension is heuristically motivated by the behaviour of perturbation theory in large orders. Two-loop running is known [40, 11] to modify the strength of renormalon singularities from poles to branch points involving the ratio β_1/β_0^2 . The class of diagrams in Fig. 5b is expected to dominate at large n over one-loop running by a factor $\beta_1/\beta_0^2 \ln n$ and is the first in a series of multiple two-loop insertions that exponentiate to produce an enhancement factor $n^{-\beta_1/\beta_0^2}$. We are thus again led by the expectation that a class of systematically large corrections can be identified and calculated. Consider a physical quantity, with perturbative expansion written as

$$R - R_{tree} = \sum_{n=0} r_n \alpha_s(Q)^{n+1}, \quad r_n = r_{n0} + r_{n1} N_f + \dots r_{nn} N_f^n, \quad (5.1)$$

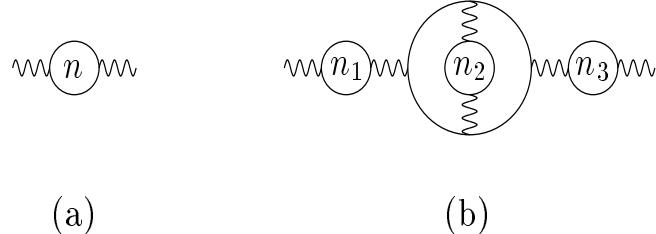


Figure 5: QED-like diagrams incorporating evolution of the coupling to leading (a) and next-to-leading order (b). A circle with letter m denotes a chain of m fermion loops. At order α^{k+1} , the relevant diagrams are specified by $n = k$ and $n_1 + n_2 + n_3 = k - 2$. At NLO, the diagrams, where the n_2 -chain forms a self-energy-type insertion, are not depicted.

assuming for simplicity that R depends only on a single scale Q , which is equal to the renormalization scale, such that the r_n are numerical coefficients.

Since the scale-setting prescription is derived from QED (that is, its fictitious version with N_f massless fermions), it is useful to consider first the abelian analogue of R . Due to the Ward identity, the evolution of the coupling in the abelian theory is generated by radiative corrections to the photon propagator. It is then natural to define the scale of the coupling at each order in perturbation theory by replacing each photon propagator $1/k^2$ by the full propagator $1/(k^2(1 + \Pi(k^2)))$ and absorbing the effect of integrating loops with full propagators into the normalization of the coupling constant to the order where the corresponding skeleton diagram appears. In this way, one is led to a modified expansion [1]

$$R - R_{tree} = r_0 \left[\alpha_s(Q^*) + \delta_1 \alpha_s(Q^{**})^2 \dots \right]. \quad (5.2)$$

Specifically, the scale of the coupling at order α_s is determined by the replacement

$$\alpha_R(Q) \rightarrow \alpha_V(k) \equiv \frac{\alpha_R(Q)}{1 + \Pi_R(k^2/Q^2, \alpha_R(Q))} \quad (5.3)$$

inside the loop-integration over k , where k is the momentum of the virtual photon. We have attached a subscript R to quantities that depend on the choice of the renormalization scheme R . On the right hand side we notice that $\alpha_V(Q)$ is the effective coupling, defined by the potential between two static sources in momentum space,

$$V(Q) = -\frac{4\pi\alpha_V(Q)}{Q^2}, \quad (5.4)$$

which defines the V-scheme [1]. Thus, in *any* scheme R , the scale Q_R^* is given by averaging the lowest order radiative corrections to the quantity R with the running coupling in the V-scheme, $\alpha_V(k)$, and not¹⁸ with $\alpha_R(k)$. By construction $\alpha_R(Q_R^*)$ is scheme-independent. The choice of $\alpha_V(k)$ to average the loop momentum distribution is physically appropriate, because the photon exchanged between static sources at distance r has momentum $k \sim 1/r$. Therefore it is in this scheme that $\alpha(k)$ can be interpreted as the effective coupling of a virtual photon of momentum k .

With this remark one is prepared to adopt the same definition in the nonabelian theory. Q^* is defined as [5]

$$r_0\alpha_R(Q^*) \equiv \int d^4k F(k, Q) \frac{\alpha_V(k)}{k^2}, \quad (5.5)$$

where $F(k, Q)$ is the integrand of the lowest order radiative correction and $C_F 4\pi \alpha_V(k)/k^2$ the potential between two static colour sources in momentum space, given by a Wilson loop. The Wilson loop is nonperturbatively well-defined (up to a distance-independent contribution) and the above integral could be evaluated without any ambiguity. In the following, we will use it only perturbatively, in the sense of an expansion of $\alpha_V(k)$ in $\alpha_V(Q)$. This restriction is not only self-imposed. Power corrections to the static potential might be different and in particular larger than those to the quantity R , in which case one would not like to use Eq. (5.5) with power-like accuracy. Let us also note that one-loop gauge boson exchange exponentiates exactly in the abelian Wilson loop but not in the nonabelian case. Therefore re-expressing $\alpha_V(Q)$ in terms of some other $\alpha_R(Q)$ inevitably involves contributions to the Wilson loop with more than one gluon exchanged.

The exact (perturbative) evaluation of Eq. (5.5) is impossible even in the abelian theory. One has to resort to some truncation of (a) the β -function in the V-scheme, needed to relate $\alpha_V(k)$ to $\alpha_V(Q)$ and (b) the relation between the coupling in the V-scheme and some other scheme R , if one wishes to express the result in terms of α_R . To leading order, the β -function has been replaced by $\beta_0 \alpha^2$. After expansion of $\alpha(k)$ in $\alpha(Q)$, this corresponds to the diagrams of Fig. 5a in QED. In this section we discuss a truncation, which is guided by the subleading N_f -dependence of coefficients and incorporates the set of diagrams depicted in Fig. 5b in addition to those in Fig. 5a. Let us first consider $n_1 = n_3 = 0$ in Fig. 5b. The diagram with $n_2 = 0$ gives $\beta_1 \alpha^3$ to the abelian β -function. For $n_2 > 0$, these diagrams give contributions to higher order coefficients to the β -function. Again in QED, this contribution can be written as $b_{n_2+1}(-\beta_1)(-\beta_0)^{n_2}$ with a N_f -independent number b_{n_2} . When expanding $\alpha_V(k)$ as

¹⁸At the level of one-loop running, the relevant transformation of schemes is simply a shift of scale, since, to this approximation,

$$\alpha_V(k) = \frac{\alpha_R(k)}{1 - \beta_0 C_R \alpha_R(k)} = \alpha_R \left(k e^{C_R/2} \right).$$

$$\alpha_V(k) = \exp \left[\ln \frac{k^2}{Q^2} \beta_V(\alpha') \frac{d}{d\alpha'} \right] \alpha'_{|\alpha'=\alpha_V(Q)}, \quad (5.6)$$

the diagrams of Fig. 5 are recovered by using

$$\beta_V(\alpha) = \beta_0 \alpha^2 + \beta_1 \sum_{n=1}^{\infty} b_n (-\beta_0)^{n-1} \alpha^{n+2}, \quad (5.7)$$

but keeping only terms with at most one power of β_1 . A different truncation could be imagined, where the β -function is kept only up to $\beta_1 \alpha^3$ but all powers of β_1 are taken into account in the expansion of Eq. (5.6). The difference to the previous truncation appears first at order α^5 .

The transition to the nonabelian theory is arguably uniquely achieved at leading order by replacing the abelian β_0 by its nonabelian value. At NLO one has to make a choice which is less compulsory: Calculate the insertions of the diagrams of Fig. 5b and express them as $c_n(-\beta_1)(-\beta_0)^{n-2}\alpha(Q)^{n+1}$ in QED. Then replace β_0 and β_1 by their nonabelian values. In effect this implies substitution of the true β -function in the V-scheme by Eq. (5.7). We can now write the perturbative coefficients of the quantity R as

$$r_n = r_{n0} + \dots r_{nn-1} N_f^{n-1} + r_{nn} N_f^n \equiv r_0 \left[\delta_n - \beta_1 (-\beta_0)^{n-2} c_n + (-\beta_0)^n d_n \right] \quad n \geq 2, \quad (5.8)$$

where, according to the prescription, the last two terms are absorbed into Q^* . Both sets of coefficients c_n and d_n are computed from abelian diagrams. Note that d_n is constructed as to eliminate the highest power of N_f . On the other hand, to NLO, the remainder δ_n still contains subleading flavour dependence, N_f^{n-1} , which arises from insertion of fermion loops into diagrams with two gluon lines as well as the effective three-gluon coupling generated by attaching three gluons to a fermion loop.

Before turning to the calculation of c_n , we want to illustrate the prescription at order α_s^3 . Up to this order, the expansion of R in a certain scheme with coupling α_s can be written as

$$R - R_{tree} = r_0 \left[\alpha_s(Q) + \{ \delta_1 + (-\beta_0)d_1 \} \alpha_s(Q)^2 + \{ \delta_2 - \beta_1 c_1 + (-\beta_0)^2 d_2 \} \alpha_s(Q)^3 \right]. \quad (5.9)$$

From Eq. (5.5) we obtain

$$\alpha_s(Q^*) = \alpha_V(Q) + (-\beta_0)d_1 \alpha_V(Q)^2 + \{ (-\beta_1)d_1 + (-\beta_0)^2 d_2 \} \alpha_V(Q)^3 \quad (5.10)$$

where the (scheme-dependent) d_n are computed as in Section 2 from diagrams with fermion loop insertions. By comparison we find that in the V-scheme ($\alpha_s(Q) = \alpha_V(Q)$) $c_1 = d_1$. The remaining linear N_f -dependence in δ_2 can then be absorbed into a scale Q^{**} of the α_s^2 -correction [7]. Notice, however, that while Q^* is defined without knowledge

of the exact α_s^3 -coefficient, the definition of Q^{**} in the nonabelian theory requires the exact α_s^3 -result.

To obtain the scale Q^* as an expansion in the coupling different from the V-scheme, one has to relate the couplings to the same accuracy and expand $\alpha_V(Q)$ in, say, $\alpha_{\overline{\text{MS}}}(Q)$ in the same form as Eq. (5.9):

$$\begin{aligned}\alpha_V(Q) &= \alpha_{\overline{\text{MS}}}(Q) + \left\{ \gamma_1 + (-\beta_0) \left(-\frac{5}{3} \right) \right\} \alpha_{\overline{\text{MS}}}(Q)^2 \\ &\quad + \left\{ \gamma_2 - \beta_1 \left(-\frac{55}{12} + 4\zeta(3) \right) + (-\beta_0)^2 \frac{25}{9} \right\} \alpha_{\overline{\text{MS}}}(Q)^3\end{aligned}\quad (5.11)$$

One can then determine c_1 in the $\overline{\text{MS}}$ scheme. Once d_1 , d_2 and c_1 are determined, Q_2^* to α_s^3 -accuracy is given by

$$Q_2^* = Q e^{-d_1/2} \left(1 + \left\{ (-\beta_0)b_1 + \tilde{b}_1 \frac{\beta_1}{\beta_0} \right\} \alpha_s(Q) \right). \quad (5.12)$$

b_1 and \tilde{b}_1 are uniquely determined by the condition

$$\begin{aligned}\alpha_s(Q_2^*) &= \alpha_s(Q) + (-\beta_0)d_1\alpha(Q)^2 + \left\{ (-\beta_0)^2(d_1^2 - 2b_1) - (d_1 - 2\tilde{b}_1)\beta_1 \right\} \alpha_s(Q)^3 + \dots \\ &\equiv \alpha_s(Q) + (-\beta_0)d_1\alpha(Q)^2 + \left\{ (-\beta_0)^2 d_2 - \beta_1 c_1 \right\} \alpha_s(Q)^3,\end{aligned}\quad (5.13)$$

which specifies that the diagrams of Fig. 5 relevant at order α_s^3 are absorbed into Q_2^* in any scheme. In the V-scheme $\tilde{b}_1 = 0$, but in general \tilde{b}_1 is non-zero.

In the V-scheme the prescription outlined here coincides exactly with the one by Brodsky and Lu [7]. When Q_2^* is defined in other schemes, we suspect that Q^* defined in [7] does not absorb the QED diagrams of Fig. 5b and part of the flavour-dependence from these diagrams is hidden in Q^{**} , since \tilde{b}_1 is always zero in [7].

When we compute c_n in the following Section, rather than using the V-scheme in intermediate steps, we will perform the average of Eq. (5.5) directly in an arbitrary scheme by replacing $\alpha_V(k)$ with the appropriately truncated form of Eq. (5.3).

5.1 Abelian diagrams at NLO

We now evaluate the perturbative coefficients generated by inserting the chains of Fig. 5 into the gluon (photon) line which appears in lowest order radiative corrections to an observable R . In this subsection β_0 and β_1 refer to their abelian values: $\beta_0 = N_f/(3\pi)$ and $\beta_1 = N_f/(4\pi^2)$. These chains have previously been considered for particular quantities: The muon anomalous magnetic moment [16] and the self-energy of a static quark [41]. Because vacuum polarization insertions are universal, the results can easily be adapted to the general case. The vacuum polarization to the present approximation is given by the diagrams with $n = 1$ and $n_1, n_3 = 0$ in Fig. 5 (Q^2 is euclidian):

$$\Pi \left(\frac{Q^2}{\mu^2}, \alpha(\mu) \right) = -\beta_0 \alpha(\mu) \left[\ln \frac{Q^2}{\mu^2} + C \right] + \beta_1 \Pi_1 \left(\frac{Q^2}{\mu^2}, \alpha(\mu) \right) \quad (5.14)$$

When $\alpha_V(k)$ is expanded inside the integrand of Eq. (5.5), the result is expressed in terms of coefficients of the vacuum polarization and averages of $\ln^n(k^2/Q^2)$ in the lowest order radiative correction $F(k, Q)$, which have already been evaluated through the insertion of the diagrams of Fig. 5a. Therefore, since Π_1 is known, the extension to incorporate Fig. 5b is merely a combinatorical problem. To organize the combinatorics, it is convenient to introduce the Borel transform. First, we define the leading order and next-to-leading order Borel transform as the result of insertion of diagrams with a single chain (Fig. 5a) and the sum of both diagrams in the Fig. 5. From Eq. (2.6) and Eq. (5.8):

$$\begin{aligned} B_{\text{LO}}[R](u) &\equiv r_0 \sum_{n=0}^{\infty} \frac{d_n}{n!} u^n \\ B_{\text{NLO}}[R](u) &\equiv r_0 \sum_{n=0}^{\infty} \frac{d_n^{\text{NLO}}}{n!} u^n \quad d_n^{\text{NLO}} = d_n - \frac{\beta_1}{\beta_0^2} c_n \quad (n \geq 2) \end{aligned} \quad (5.15)$$

and $d_n^{\text{NLO}} = d_n$ for $n = 0, 1$. After Borel transformation of Eq. (5.5), the truncation of $\alpha_V(k)$ discussed above corresponds to insertion of

$$B \left[\frac{\alpha(Q)}{1 + \Pi(k^2/Q^2, \alpha(Q))} \right] (u) = \left(\frac{k^2}{Q^2} e^C \right)^{-u} - \frac{\beta_1}{\beta_0^2} \int_0^u dv v \left(\frac{k^2}{Q^2} e^C \right)^{-v} B \left[\frac{\Pi_1}{\alpha} \right] (u-v) \quad (5.16)$$

into Eq. (5.5) instead of the complete $\alpha_V(k)$. On the right hand side, we have neglected multiple insertions of Π_1 . The Borel transform of Π_1/α can be represented as

$$B \left[\frac{\Pi_1}{\alpha} \right] (u) = \left(\frac{k^2}{Q^2} \right)^{-u} F(u) - G(u), \quad (5.17)$$

where $F(u)$ is a scheme-independent function that can be obtained from Eq. (3.9) by integration with respect to Q^2 :

$$F(u) = \frac{32}{3} \frac{1}{1 - (1-u)^2} \sum_{k=2}^{\infty} \frac{(-1)^k k}{(k^2 - (1-u)^2)^2} \equiv \sum_{n=-1}^{\infty} f_n u^n \quad (5.18)$$

$G(u)$ is a scheme-dependent integration constant. One can use the renormalization group equation obeyed by the photon vacuum polarization to relate the expansion coefficients of $G(u)$ to those of the β -function in the chosen scheme. The precise relation is as follows: Let us write the highest power of β_n as

$$\beta_n|_{N_f^n} \equiv b_n \beta_1 (-\beta_0)^{n-1}. \quad (5.19)$$

Then

$$G(u) = \sum_{n=0}^{\infty} \frac{b_{n+1}}{n!} u^{n-1}, \quad g_n = \frac{b_{n+2}}{(n+1)!}. \quad (5.20)$$

With these preliminaries, we insert Eq. (5.16) into Eq. (5.5) and obtain

$$\begin{aligned} B_{\text{NLO}}[R](u) &= B_{\text{LO}}[R](u) - \frac{\beta_1}{\beta_0^2} \int_0^u \frac{dv v}{u-v} (B_{\text{LO}}[R](u) - B_{\text{LO}}[R](v)) \\ &\quad - \frac{\beta_1}{\beta_0^2} \int_0^u dv v (B_{\text{LO}}[R](u) F_{\text{reg}}(u-v) - B_{\text{LO}}[R](v) G_{\text{reg}}(u-v)), \end{aligned} \quad (5.21)$$

where $F_{\text{reg}}(u)$ and $G_{\text{reg}}(u)$ are defined by removing the pole at $u = 0$: $F_{\text{reg}}(u) \equiv F(u) - 1/u$, $G_{\text{reg}}(u) \equiv G(u) - 1/u$.

The coefficients d_n^{NLO} are given by

$$d_n^{\text{NLO}} = \frac{1}{r_0} \frac{d^n}{du^n} B_{\text{NLO}}[R](u) \Big|_{u=0}. \quad (5.22)$$

Taking the derivatives of Eq. (5.21) we obtain (for $n \geq 2$)

$$\begin{aligned} d_n^{\text{NLO}} &= d_n - \frac{\beta_1}{\beta_0^2} n (\psi(n+1) - \psi(2)) d_{n-1} \\ &\quad - \frac{\beta_1}{\beta_0^2} \sum_{k=0}^{n-2} \left[\binom{n}{k} f_{n-2-k} - (k+1) g_{n-2-k} \right] (n-2-k)! d_k, \end{aligned} \quad (5.23)$$

where $\psi(x)$ is the logarithmic derivative of the Γ -function. Note that for large n the second term on the right side indeed dominates the first one by a factor $\beta_1/\beta_0^2 \ln n$. With correspondingly changed conventions, this equation agrees with the corresponding one in [16]. The expansion coefficients of $F(u)$ are given by [16]

$$f_n = -\frac{2}{3}(n+2) \left[-2n - 2 - \frac{n+7}{2^{n+3}} + \frac{16}{n+2} \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor} k(1-2^{-2k})(1-2^{2k-n-3})\zeta(2k+1) \right], \quad (5.24)$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ and $[..]$ denotes the integer part of the number in brackets. The coefficients g_n depend on the scheme employed for the definition of α . In the $\overline{\text{MS}}$ scheme, we can use the β -function to the approximation required here from [39, 16]. We then find

$$g_n^{\overline{\text{MS}}} = \frac{1}{(n+1)!(n+2)!} \frac{d^{n+1}}{du^{n+1}} \left[\frac{(1-u)(1+2u)(3+2u)\Gamma(4+2u)}{9\Gamma(2+u)^2\Gamma(3+u)\Gamma(1-u)} \right] \Big|_{u=0}. \quad (5.25)$$

In the V -scheme, we have

$$g_n^V = f_n. \quad (5.26)$$

The simplest way to see this is that in this scheme by definition $\Pi(1, \alpha_V(Q)) = 0$. Then the expansion of the integrand in Eq. (5.5) has only non-zero powers of $\ln(k^2/Q^2)$ and d_0 can not appear in Eq. (5.23).

5.2 Numerical analysis

The transition to QCD according to the prescription formulated above is performed by replacing β_1/β_0^2 in Eq. (5.23) by its QCD value. Note that contrary to d_n the NLO coefficient depends explicitly on the number of flavours. In this Subsection, rather than presenting values for the BLM scale Q^* to NLO accuracy, we give the perturbative coefficients in low orders generated by expansion of $\alpha_s(Q^*)$ in the $\overline{\text{MS}}$ -coupling, that is the coefficients d_n^{NLO} and $M_n^{\text{NLO}}(-\beta_0\alpha_s)$, defined in complete analogy with $M_n(-\beta_0\alpha_s)$. To compare the coefficients obtained from keeping only vacuum polarization with the exact ones, if available, we shall also define d_n^{exact} as the exact coefficient, divided by $(-\beta_0)^n$. Numerical values for the difference between pole and $\overline{\text{MS}}$ mass are shown in Table 5, and for the derivative of the hadronic vacuum polarization $Q^2 d\Pi/dQ^2$ and the τ decay width in Tables 6 and 7.

The main conclusion for the radiative mass shift is that the effect of two-loop running remains small up to the order, at which the series has to be truncated. It leads to a less than 5% additional increase of the one-loop radiative correction, which can be neglected in view of the uncertainties that have been discussed in Sect. 4. This is reassuring, because it supports the suggestion that the mass shift is to a large extent given by one-loop running effects, although it must be borne in mind that by incorporating two-loop running one does not gain control over genuine two- and higher loop corrections.

On the other hand, the effect of two-loop running is very large for the derivative of the vacuum polarization and the efficiency of extending the BLM prescription beyond leading order can be doubted in this case. Although the inclusion of two-loop running improves the estimate of d_2 as compared to the exact coefficient d_2 (Table 6), for $n > 3$ the effect of two-loop running is exceedingly large and this improvement might as well be accidental.

It is interesting to observe that for the τ hadronic width itself (Table 7), such irregularities do not occur, which is in qualitative agreement with the conclusion of Sect. 3.1 that the perturbative expansion of R_τ has a smoother behaviour in intermediate orders of perturbation theory than $Q^2 d\Pi/dQ^2$. The relative importance of two-loop running increases with n as expected from the asymptotic $\ln n$ -enhancement. Though smooth, the effect of two-loop running is significant for R_τ and points towards an even further increase of the cumulative effect of higher order perturbative corrections and consequently a further decrease of $\alpha_s(m_\tau)$ from R_τ .

n	d_n	d_n^{NLO}	M_n [b-quark]	M_n^{NLO} [b-quark]
0	1	1	1	1
1	4.68615	4.68615	1.623	1.623
2	17.6227	19.6884	1.935	1.972
3	109.859	127.529	2.193	2.272
4	873.924	1138.28	2.467	2.628
5	8839.69	12085.6	2.835	3.131

Table 5: Comparison of leading order and next-to-leading order coefficients for the difference of pole and $\overline{\text{MS}}$ mass, where quark masses inside loops are neglected and loops containing the heavy quark are excluded (cf. Sect. 4). The comparison of the modification of lowest order radiative corrections is for $\alpha_s(m_b) = 0.2$, relevant to bottom quarks.

n	d_n	d_n^{NLO}	d_n^{exact}	M_n	M_n^{NLO}	M_n^{exact}
0	1	1	1	1	1	1
1	0.691772	0.691772	0.728809	1.158	1.158	1.167
2	3.10345	2.31417	1.25847	1.321	1.271	1.233
3	2.18004	7.40378	—	1.347	1.360	—
4	30.7398	18.4580	—	1.432	1.411	—
5	-34.5336	146.293	—	1.410	1.503	—

Table 6: Comparison of leading order, next-to-leading order and exact coefficients of $Q^2 d\Pi(Q)/dQ^2$. The NLO and exact values are given for $N_f = 3$ and the partial sums M_n for $\alpha_s = 0.32$. Exact values, when available, include genuine higher order corrections.

n	d_n	d_n^{NLO}	d_n^{exact}	M_n	M_n^{NLO}	M_n^{exact}
0	1	1	1	1	1	1
1	2.27511	2.27511	2.31213	1.521	1.521	1.529
2	5.68475	5.98780	5.20806	1.819	1.835	1.803
3	13.7536	18.1248	—	1.984	2.053	—
4	35.1470	54.7939	—	2.081	2.203	—
5	84.4066	178.897	—	2.134	2.316	—

Table 7: Same as in the previous table for the perturbative expansion of the τ hadronic width ($\alpha_s(m_\tau) = 0.32$).

In Appendix C, we give exact results for the five-loop diagrams that enter the calculation of d_3^{NLO} for $Q^2 d\Pi/dQ^2$.

6 Conclusions

In this paper we have shown how to deal with higher order vacuum polarization insertions into radiative corrections to observables in QCD. This enterprise has been motivated by the fact that these higher order corrections often lead to disturbingly large coefficients in the perturbative expansion. This trend is systematic, but since it can be identified, we suggest to calculate these corrections and separate them from the remaining ones. The behaviour of the remaining series should then be improved. Even if the remaining corrections turn out not to be small, the summation of vacuum polarization insertions can be motivated as a physical way of scale-setting. In this respect it can be considered as an extension of the scale-setting proposed by Brodsky, Lepage and Mackenzie [1] and Brodsky and Lu [7], as far as two-loop running is concerned. It is worth noting that the familiar BLM scale typically overestimates the size of higher order vacuum polarization insertions. The smallness of this scale needs not necessarily indicate a failure of perturbation theory, unless it is indicative of its divergence already at two-loop order.

We would like to stress the computational ease with which the summation of effects of one-loop running can be performed compared to a genuine higher order calculation. We thus have devoted considerable space to technical aspects, which should allow routine implementation of this summation. The two existing techniques can be summarized as follows: The first one [14, 16, 17, 3] amounts to direct evaluation of the relevant higher order Feynman diagrams. A convenient tool to organize such a calculation is the Borel transform. The result can be obtained as a certain analytically regularized Feynman integral plus – if necessary – a subtraction function that is easily computed directly from a large-mass expansion. This technique has its limits, when applied to processes with several scales or to physical cross sections that are more directly calculated as a sum of real and virtual corrections. In these cases, we have applied a dispersion technique [20, 12, 4] (or, if one starts from the Borel transform, a Mellin transformation) to reduce the problem to calculation of lowest order corrections with finite gluon mass. It is in this representation, that summation (up to an accuracy set by renormalons) is most easily performed. This summation is usually complicated by the need to analytically continue the Borel transform beyond its radius of convergence and to take the (principal value) integral over the Borel parameter. Our Eqs. (2.24) and (2.29) are immediately suited to numerical evaluation. Given this simplicity, the computational expense seems worth the investment even if one could only hope to absorb a higher order radiative correction that has the correct sign compared to the exact one.

In the present paper, we have investigated two quantities in more detail. Higher order $\beta_0^n \alpha_s^{n+1}$ -corrections are indeed sizeable for the hadronic decay width of the τ lepton, when expressed as a series in α_s . We compared fixed order perturbative approximations to R_τ with those from a partial resummation of running coupling effects due to contour

integration [10] and found that in intermediate orders the first one has a smoother behaviour, since potentially large corrections from contour integration conspire with the divergent coefficients of $Q^2 d\Pi/dQ^2$ to produce an effective suppression of ultraviolet renormalons (Table 1). Resummation of all one-loop running coupling effects leads to an approximate 10% decrease of $\alpha_s(m_\tau)$ determined from hadronic τ decays to a central value of¹⁹

$$\alpha_s(m_\tau) \simeq 0.29. \quad (6.1)$$

This shift of about one error margin of previous analyses is not caused by a single large next-order coefficient, but by the constructive addition of several higher order coefficients beyond the exactly known coefficient at order α_s^3 .

The accurate extraction of α_s from τ -decays relies on the absence of $\Lambda_{\text{QCD}}^2/m_\tau^2$ -corrections. Although from a purely theoretical point of view, the situation with respect to such terms is not conclusive, given that the limitations of duality are only poorly understood, we conclude that effects associated with summation of the divergent perturbative expansion should be excluded as a source of such terms, provided one accepts their absence in the relevant current correlation functions in euclidian space.

For the difference between the pole and $\overline{\text{MS}}$ -renormalized mass of a heavy quark, we conclude that the large two-loop correction found in [38] is probably not accidental but the first one in a rapidly divergent series of higher order corrections dominated indeed by one-loop running of the coupling. For the charm quark, the divergence is such that the perturbative series can certainly not be improved beyond two-loop order. For the bottom pole mass, our estimate incorporating one-loop running reads

$$\frac{\delta M_b}{M_b} = (16.3 \pm 2.9 \pm 1.5)\%, \quad (6.2)$$

which is about 30% larger than the two-loop result without resummation. We expect this difference to be significant in phenomenological applications. The first error quoted is associated with the divergence of the series that relates the two mass definitions and is irreducible. Numerically this error amounts to an uncertainty somewhat larger than 100 MeV for the mass difference, a value in between the estimates initially reported in [37, 17]. Short-distance observables containing quark masses, when expressed in terms of pole masses will generally exhibit large coefficients implanted by the use of this parameter. Thus, although both mass definitions are ultimately unphysical, we expect that the $\overline{\text{MS}}$ mass can be determined to better accuracy through perturbative relations²⁰.

Finally, let us mention that the calculation of vacuum polarization can never replace an exact calculation. Until this is completed, we find it worthwhile to incorporate the systematic effects exhibited by one-loop running. Surely, the authors are among those

¹⁹As an input we have taken $R_\tau = 3.56 \pm 0.03$ from [22], see Sect. 3.1.

²⁰This argument does not apply to pole mass differences. The large coefficients associated with the pole mass can be understood as a universal (flavour-independent) additive mass renormalization, which cancels in the difference.

who await further exact results with suspense.

Note added. While this paper was written, we were informed by M. Neubert on work of his, which partially overlaps with the present one. We acknowledge exchange of manuscripts and are very grateful for the ensuing discussions which helped to clarify our presentation. We disagree with the conclusion of [42] that the difference of various summation prescriptions for cross sections should be interpreted as a sign and quantification of the failure of the operator product expansion in the physical region. As emphasized in Sect. 3.2, once one abandones Borel-summation type prescriptions, which are distinguished by their relation to the OPE, one can introduce $1/Q^2$ -differences in euclidian space just as well and there is no discrimination between the euclidian and minkowskian region from the point of view of summation of perturbation theory, given the present state of knowledge.

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Appendices

A Subtractions

In general, we will also be interested in ultraviolet divergent quantities. Examples include the difference between the pole mass and the $\overline{\text{MS}}$ -renormalized quark mass and the correlation function of heavy-light currents in heavy quark effective theory. Denoting a generic quantity by $R(\alpha)$, the renormalized Borel transform has the form

$$B[R](u) = B[R]_0(u) + S_R(u), \quad (\text{A.1})$$

where $B[R]_0(u)$ is the bare Borel transform, calculated from diagrams without overall subtraction, using the Borel parameter itself as a regulator. This corresponds to analytic regularization [15]. Consequently, the bare Borel transform is singular at the origin (Recall that the perturbative series is generated by derivatives of the Borel transform at the origin.). This singularity is subtracted by the function $S_R(u)$, which depends on the renormalization scheme. In minimal subtraction schemes $uS_R(u)$ is an entire function (at least for one chain of fermion loops) and – when combined with the appropriate singular piece of $B[R]_0(u)$ to be defined below, Eq. (A.17) – $S_R(u)$ yields an unambiguous contribution to the bubble sum

$$S_R(\alpha) \equiv \left(-\frac{1}{\beta_0}\right) \int_0^\infty du e^{-u/(-\beta_0\alpha)} \left(S_R(u) + \text{singular term in } B[R]_0(u)\right). \quad (\text{A.2})$$

General expressions for $S_R(u)$ in minimal subtraction schemes have been derived in Appendix A of [17]. In this Appendix we outline a simple method to obtain $S_R(u)$ and calculate $S_R(\alpha)$ in minimal subtraction schemes. An essential simplification comes from the fact that one needs only the dominant large- λ behaviour of dimensionally regularized Feynman amplitudes, where λ is a mass for the gluon. As an illustration of the technique we calculate the subtractions for the correlation function of two heavy-light currents in heavy quark effective theory.

A.1 Calculation of $S_R(\alpha)$

As always we assume that the quantity R has been made dimensionless, is infrared finite and such that the order- α radiative correction comes from gluon exchange. We shall also assume that the corresponding diagrams have one loop and are at most logarithmically ultraviolet divergent. The last assumption is actually unnecessary, see the explicit example in the subsequent subsection. The dimensionally regularized order- α correction can then be represented as

$$r_0^{\text{bare}}(\epsilon) \alpha = \alpha \mu^{2\epsilon} \int d^d k F(k, Q) \frac{1}{k^2}. \quad (\text{A.3})$$

We use $d = 4 - 2\epsilon$ and denote by Q a set of external momenta. For gauge-dependent quantities, we take Landau gauge. The factor $(k_\mu k_\nu/k^2 - g_{\mu\nu})$ from the gluon propagator is included in the integrand $F(k, Q)$ and we do not write Lorentz indices explicitly. We define the both, dimensionally and analytically regularized coefficient by

$$\hat{r}_0^{\text{bare}}(s, \epsilon) \alpha = \alpha \mu^{2\epsilon} \int d^d k F(k, Q) \frac{1}{k^2} \left(-\frac{\mu^2}{k^2} \right)^s, \quad (\text{A.4})$$

where μ is a subtraction scale. Note that $B[R]_0(u) = \hat{r}_0^{\text{bare}}(u, \epsilon = 0) e^{-uC}$, see Eq. (2.12).

Insert n fermion loops into the gluon line of all diagrams that generate r_0 . The fermion loop integrations can be done. Each loop gives a factor

$$-\frac{\beta_0^f}{\epsilon} \frac{6\Gamma(1+\epsilon)\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} \left(-\frac{k^2}{4\pi\mu^2} \right)^{-\epsilon},$$

where $\beta_0^f = T/(3\pi)$ is the fermionic contribution to the beta-function ($T = 1$ in QED and $T = 1/2$ in QCD). Performing the final integration over gluon momentum k , the result for the coefficient of order α^{n+1} can be written as

$$r_n^{\text{bare}}(\epsilon) = \frac{(\beta_0^f)^n}{(n+1)(-\epsilon)^{n+1}} G(-\epsilon, -(n+1)\epsilon) \quad (\text{A.5})$$

$$G(-\epsilon, -(n+1)\epsilon) = \left((4\pi)^\epsilon \frac{6\Gamma(1+\epsilon)\Gamma(2-\epsilon)^2}{\Gamma(4-2\epsilon)} \right)^n (n+1)(-\epsilon) \hat{r}_0^{\text{bare}}(s = n\epsilon, \epsilon), \quad (\text{A.6})$$

where Eq. (A.4) has been used. The function G introduced above coincides with the one of Appendix A of [17] [Note that $d = 4 + 2\epsilon$ has been used in [17], which motivates the signs in the arguments of G above.]. To obtain the subtraction function $S_R(u)$ in the $\overline{\text{MS}}$ scheme, one has to add to $r_n^{\text{bare}}(\epsilon)$ the diagrams with fermion loops replaced by their $\overline{\text{MS}}$ counterterms and then compute the finite part of the sum. For any quantity R , the steps of Appendix A in [17] for the self-energy of a heavy quark can be repeated with the general result

$$S_R(u) = \frac{\tilde{G}_0(u)}{u} \quad \tilde{G}_0(u) = \sum_{n=0}^{\infty} \frac{g_n}{n!} u^n, \quad (\text{A.7})$$

where g_n are the expansion coefficients of

$$G_0(-\epsilon) \equiv G(-\epsilon, -(n+1)\epsilon = 0) = \sum_{m=0}^{\infty} g_m (-\epsilon)^m. \quad (\text{A.8})$$

Since G_0 is related to subtractions, it should be unnecessary to calculate the full diagram, encoded in the function G , in order to obtain G_0 . We prove that G_0 can be deduced from the large-mass expansion of the dimensionally regularized one-loop coefficient $r_0^{\text{bare}}(\lambda^2, \epsilon)$ computed with a gluon mass λ . That is, $1/k^2$ in Eq. (A.3) is replaced

by $1/(k^2 - \lambda^2)$ (in Landau or Feynman gauge). Since the original four-dimensional integral was logarithmically ultraviolet divergent, the asymptotic behaviour at large λ is given by

$$r_0^{\text{bare}}(\lambda^2, \epsilon) \xrightarrow{\lambda^2 \rightarrow \infty} -\frac{1}{\epsilon} r_\infty(\epsilon) \left(\frac{\mu^2}{\lambda^2} \right)^\epsilon. \quad (\text{A.9})$$

This equation defines the λ^2 -independent function $r_\infty(\epsilon)$. Its value at $\epsilon = 0$ is denoted by r_∞ . By definition

$$G_0(-\epsilon) = -\frac{1}{(4\pi)^\epsilon} \frac{\Gamma(4-2\epsilon)}{6\Gamma(1+\epsilon)\Gamma(2-\epsilon)^2} \lim_{s \rightarrow -\epsilon} (s+\epsilon) \hat{r}_0^{\text{bare}}(s, \epsilon). \quad (\text{A.10})$$

To evaluate the limit, we use that $\hat{r}_0^{\text{bare}}(s, \epsilon)$ is related to the one-loop coefficient with finite gluon mass $r_0^{\text{bare}}(\lambda^2, \epsilon)$ by a Mellin transform:

$$\hat{r}_0^{\text{bare}}(s, \epsilon) = -\frac{\sin \pi s}{\pi} \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left(\frac{\lambda^2}{\mu^2} \right)^{-s} r_0^{\text{bare}}(\lambda^2, \epsilon) \quad (\text{A.11})$$

This represents the analytic continuation of the analogon of Eq. (2.17) into the u -interval $] -\epsilon, 0[$. Integrating by parts, we get

$$\hat{r}_0^{\text{bare}}(s, \epsilon) = -\frac{\sin \pi s}{\pi(s+\epsilon)} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} \right)^{-(s+\epsilon)} f'(\lambda^2, \epsilon), \quad (\text{A.12})$$

where $f(\lambda^2, \epsilon)$ is defined by dividing $\hat{r}_0^{\text{bare}}(s, \epsilon)$ by $(\mu^2/\lambda^2)^\epsilon$. The limit is now easily taken with the result

$$\lim_{s \rightarrow -\epsilon} (s+\epsilon) \hat{r}_0^{\text{bare}}(s, \epsilon) = -\frac{\sin \pi \epsilon}{\pi \epsilon} r_\infty(\epsilon). \quad (\text{A.13})$$

This yields the final result

$$G_0(u) = \frac{1}{(4\pi)^{-u}} \frac{\Gamma(4+2u)}{6\Gamma(1-u)\Gamma(2+u)^2} \frac{\sin \pi u}{\pi u} r_\infty(-u). \quad (\text{A.14})$$

The computation of the subtraction function has been reduced to calculation of the large mass limit of one-loop corrections with non-zero gluon mass.

Combining Eq. (2.17) for the bare Borel transform with Eq. (A.7), the renormalized Borel transform is given by

$$B[R](u) = -\frac{\sin \pi u}{\pi u} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} r'_0(\lambda^2) + \frac{\tilde{G}_0(u)}{u}. \quad (\text{A.15})$$

The derivative $r'_0(\lambda^2)$ is ultraviolet finite, which allows to put $\epsilon = 0$. However, the integral is not yet finite at $u = 0$ and the above expression is not suited to take derivatives at $u = 0$. Since, by Eq. (A.9), $r'_0(\lambda^2) = r_\infty/\lambda^2$ for large λ^2 , we obtain

$$\begin{aligned}
B[R](u) = & -\frac{\sin \pi u}{\pi u} \int_0^\infty d\lambda^2 \left(\frac{\lambda^2}{\mu^2} e^C \right)^{-u} \left[r'_0(\lambda^2) - \frac{r_\infty}{\lambda^2} \Theta(\lambda^2 - \mu^2 e^{-C}) \right] \\
& + \frac{1}{u} \left(\tilde{G}_0(u) - r_\infty \frac{\sin \pi u}{\pi u} \right). \tag{A.16}
\end{aligned}$$

In this form the integral exists for $u = 0$ and the second line is finite at $u = 0$. We note that this expression is equivalent to, but slightly different from the one given in [4]. In the present form μ^2 and C appear only in their natural combination $\mu^2 e^{-C}$.

To obtain the bubble sum, all steps that lead from Eq. (2.17) to Eq. (2.24) can now be repeated on the first line of Eq. (A.16). The Borel integral of the second line is given by

$$\begin{aligned}
S_R(\alpha) \equiv & \left(-\frac{1}{\beta_0} \right) \int_0^\infty du e^{-u/(-\beta_0 \alpha)} \frac{1}{u} \left(\tilde{G}_0(u) - r_\infty \frac{\sin \pi u}{\pi u} e^{-uC} \right) \tag{A.17} \\
= & \left(-\frac{1}{\beta_0} \right) \int_0^{-\beta_0 \alpha} \frac{du}{u} (G_0(u) - r_\infty) - \frac{r_\infty}{\beta_0} \left[\frac{\arctan(\pi \beta_0 \alpha)}{\pi \beta_0 \alpha} + \frac{1}{2} \ln \left(1 + \pi^2 \beta_0^2 \alpha^2 \right) - 1 \right].
\end{aligned}$$

Thus, in case subtractions are required beyond coupling renormalization, Eq. (2.24) is replaced by

$$\begin{aligned}
r_0 a_s M_\infty(a_s) = & \int_0^\infty d\lambda^2 \Phi(\lambda^2) \left(r'_0(\lambda^2) - \frac{r_\infty}{\lambda^2} \Theta(\lambda^2 - \mu^2 e^{-C}) \right) + [r_0(\lambda_L^2) - r_0(0)] \\
& + \int_0^{a_s} \frac{du}{u} (G_0(u) - r_\infty) + r_\infty \left[\frac{\arctan(\pi a_s)}{\pi a_s} + \frac{1}{2} \ln \left(1 + \pi^2 a_s^2 \right) - 1 \right], \tag{A.18}
\end{aligned}$$

with $G_0(u)$ given by Eq. (A.14).

A.2 A sample calculation

As a non-trivial example, we calculate the subtraction function for the correlation function of heavy-light currents in heavy quark effective theory,

$$\Pi_5(\omega) = i \int d^4x e^{i\omega(v \cdot x)} \langle 0 | T\{j_5^\dagger(x) j_5(0)\} | 0 \rangle \quad j_5(x) = \bar{h}_v(x) i\gamma_5 q(x). \tag{A.19}$$

The corresponding bare Borel transform has been given in [17]. At first sight the method exposed in the previous subsection appears inapplicable, because the diagrams to be considered have two loops, see Fig. 6, and, because the correlation function is quadratically divergent, one also needs subtractions for the two-loop diagram itself. To eliminate these, we shall consider the third derivative

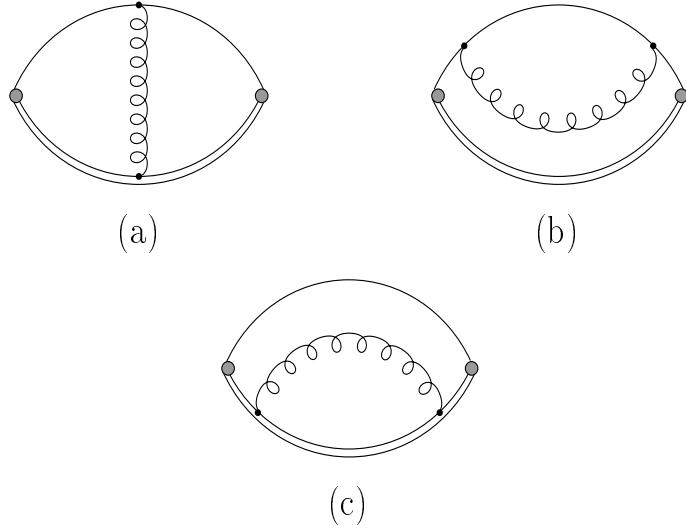


Figure 6: Radiative corrections to the correlation function of heavy-light currents. The shaded circle represents a current insertion with momentum q and the double line denotes a heavy quark propagator.

$$D(\omega) \equiv \omega \frac{d^3\Pi_5(\omega)}{d\omega^3}. \quad (\text{A.20})$$

Then all subtractions originate from divergent one-loop subdiagrams, which are subsequently inserted in the lowest order one-loop diagram for $\Pi_5(\omega)$. We remind the reader that the heavy-light current in the effective theory is not conserved. We treat the three diagrams in turn and take Landau gauge.

Let us start with diagram (a) and investigate the one-loop correction to the heavy-light vertex with non-zero gluon mass. The numerator of the corresponding Feynman integrand is proportional to

$$\gamma^\rho(\not{q} - \not{k})\gamma_5 v^\tau(k^2 g_{\rho\tau} - k_\rho k_\tau), \quad (\text{A.21})$$

where v is the velocity of the heavy quark and the momentum assignments to the lines of the diagram are evident. Since the contribution proportional to $\lambda^{-2\epsilon}$ required by Eq. (A.9) comes from $k \sim \lambda \gg q$, we can drop \not{q} in the above expression and the integral is simplified to

$$\int \frac{d^d k}{(2\pi)^d} \frac{\not{v} \not{k} - v \cdot k}{(q - k)^2 v \cdot (q' - k) (k^2 - \lambda^2)}. \quad (\text{A.22})$$

The corresponding integral with numerator replaced by k_μ has structure $F_1 v_\mu + F_2 q_\mu + F_3 q'_\mu$. By dimensional counting, only F_1 can behave as $\lambda^{-2\epsilon}$ for large λ^2 , the other structures being suppressed by powers of λ . But then the potential term $\lambda^{-2\epsilon}$ is proportional to $\not{v} - v^2 = 0$ and therefore $r_\infty(\epsilon)$ vanishes identically for this diagram (This property is specific to Landau gauge.).

For diagram (b) we need the self-energy insertion of a light (massless) quark, given by

$$- C_F g^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\rho (\not{p} - \not{k}) \gamma^\tau}{(p - k)^2 (k^2 - \lambda^2)} \left[g_{\rho\tau} - \frac{k_\rho k_\tau}{k^2} \right], \quad (\text{A.23})$$

where p is the external momentum for the subdiagram. Again the term involving $\lambda^{-2\epsilon}$ in the large λ^2 -expansion comes from $k \sim \lambda \gg p$, which allows to expand the propagator $1/(p - k)^2$. The integrand simplifies to

$$\frac{\gamma^\rho (\not{p} - \not{k}) \gamma^\tau}{(p - k)^2 (k^2 - \lambda^2)} \left[g_{\rho\tau} - \frac{k_\rho k_\tau}{k^2} \right] \longrightarrow \frac{\epsilon (3 - 2\epsilon)}{2 - \epsilon} \frac{\not{p}}{k^2 (k^2 - \lambda^2)} + O\left(\frac{1}{k^5}\right), \quad (\text{A.24})$$

where the neglected terms do not produce a $\lambda^{-2\epsilon}$ contribution. The result is

$$(-i) \not{p} \left(\frac{4\pi\mu^2}{\lambda^2} \right)^\epsilon \frac{C_F \alpha}{4\pi} \frac{(3 - 2\epsilon)\Gamma(1 + \epsilon)}{(1 - \epsilon)(2 - \epsilon)}. \quad (\text{A.25})$$

Integration of the final quark loop is straightforward, and taking three subtractions we obtain

$$G_0^{(b)}(u) = \frac{C_F N_c}{4\pi^3} \left[-\frac{u}{6} (3 + 2u) \frac{\Gamma(4 + 2u)}{\Gamma(1 - u)\Gamma(2 + u)^2\Gamma(3 + u)} \right]. \quad (\text{A.26})$$

Turning to the self-energy of a heavy quark in heavy quark effective theory in diagram (c), we encounter another apparent obstacle to applying the technique of the previous subsection: The subdiagram is linearly ultraviolet divergent, and – at large λ^2 – behaves like $\lambda^{1-2\epsilon}$. The remedy is simply to ignore this term and pick up the term $\lambda^{-2\epsilon}$, since power divergences are discarded in minimal subtraction schemes. The large mass expansion of the heavy quark self-energy is

$$(-i) \frac{1 + \not{p}}{2} \left(\frac{4\pi\mu^2}{\lambda^2} \right)^\epsilon \frac{C_F \alpha}{4\pi} \left[\Gamma\left(\frac{1}{2}\right) \Gamma\left(\epsilon - \frac{1}{2}\right) \lambda + \frac{(3 - 2\epsilon)\Gamma(\epsilon)}{1 - \epsilon} v \cdot p + O\left(\frac{v \cdot p}{\lambda}\right) \right]. \quad (\text{A.27})$$

Discarding the first term and inserting the remaining one into the quark loop, we get

$$G_0^{(c)}(u) = \frac{C_F N_c}{4\pi^3} \left[\frac{1}{6} (3 + 2u) \frac{\Gamma(4 + 2u)}{\Gamma(1 - u)\Gamma(2 + u)^3} \right]. \quad (\text{A.28})$$

Adding the contributions from all diagrams, the result is

$$G_0(u) = \frac{C_F N_c}{4\pi^3} \left[\frac{1}{3}(3+2u) \frac{\Gamma(4+2u)}{\Gamma(1-u)\Gamma(2+u)^2\Gamma(3+u)} \right]. \quad (\text{A.29})$$

B Radiative corrections to $R_{e^+e^-}$ and R_τ with finite gluon mass

We write the lowest order radiative corrections to the total cross section of e^+e^- annihilation and the hadronic τ decay width with finite gluon mass λ as

$$R_{e^+e^-} = 3 \left[1 + \alpha_s(s) \left\{ r_{e^+e^-}^{virt}(y) + \Theta(1-y) r_{e^+e^-}^{real}(y) \right\} \right], \quad (\text{B.1})$$

$$R_\tau = 3 \left[1 + \alpha_s(m_\tau) \left\{ r_\tau^{virt}(y) + \Theta(1-y) r_\tau^{real}(y) \right\} \right], \quad (\text{B.2})$$

where $y = \lambda^2/s$ or $y = \lambda^2/m_\tau^2$, respectively. Quarks are taken as massless. $r^{virt}(y)$ are virtual and $r^{real}(y)$ real gluon emission corrections:

$$\begin{aligned} r_{e^+e^-}^{virt}(y) &= \frac{2}{3\pi} \left[2(1+y)^2 \left\{ -\frac{1}{2} \ln^2 y + \ln y \ln(1+y) + \frac{\pi^2}{6} + \text{Li}_2(-y) \right\} \right. \\ &\quad \left. - \frac{7}{2} - 2y - 3 \ln y - 2y \ln y \right] \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} r_{e^+e^-}^{real}(y) &= \frac{2}{3\pi} \left[2(1+y)^2 \left\{ \frac{1}{2} \ln^2 y - 2 \ln y \ln(1+y) - \frac{\pi^2}{6} - 2\text{Li}_2(-y) \right\} \right. \\ &\quad \left. + 5 - 5y^2 + 3 \ln y + 4y \ln y + 3y^2 \ln y \right], \end{aligned} \quad (\text{B.4})$$

and

$$\begin{aligned} r_\tau^{virt}(y) &= \frac{1}{324\pi} \left[-2577 + 72\pi^2 + (2392 - 240\pi^2)y + (828 - 432\pi^2)y^2 + 144(1 - \pi^2)y^3 \right. \\ &\quad - 24\pi^2y^4 - 1332 \ln y + (2208 + 288\pi^2)y \ln y + 792y^2 \ln y + 144y^3 \ln y - 216 \ln^2 y \\ &\quad + 720y \ln^2 y + 1296y^2 \ln^2 y + 432y^3 \ln^2 y + 72y^4 \ln^2 y + 864y \ln^3 y \\ &\quad + (432 - 1440y - 2592y^2 - 864y^3 - 144y^4)(\ln y \ln(1+y) + \text{Li}_2(-y)) \\ &\quad \left. + 1728y \ln y \text{Li}_2(-y) - 3456y \text{Li}_3\left(-\frac{1}{y}\right) \right] \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} r_\tau^{real}(y) &= \frac{1}{324\pi} \left[2901 - 72\pi^2 + (-9736 + 240\pi^2 + 5184\zeta(3))y + (6120 + 432\pi^2)y^2 \right. \\ &\quad + (456 + 144\pi^2)y^3 + (259 + 24\pi^2)y^4 + 1332 \ln y - (1776 + 864\pi^2)y \ln y - 4176y^2 \ln y \end{aligned}$$

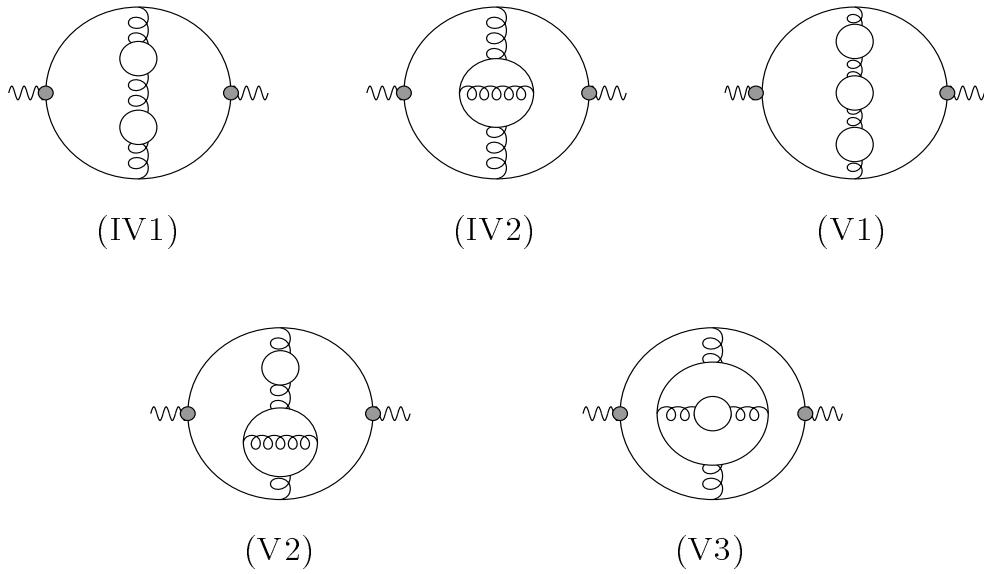


Figure 7: Representative abelian four- and five-loop diagrams for the hadronic vacuum polarization. It is understood that all self-energy type diagrams are added to each class of diagrams.

$$\begin{aligned}
& -288y^3 \ln y + 216 \ln^2 y - 720y \ln^2 y - 1296y^2 \ln^2 y - 864y^3 \ln^2 y - 144y^4 \ln^2 y \\
& -1440y \ln^3 y + (-864 + 2880y + 5184y^2 + 1728y^3 + 288y^4) (\ln y \ln(1+y) + \text{Li}_2(-y)) \\
& -3456y \ln y \text{Li}_2(-y) + 6912y \text{Li}_3\left(-\frac{1}{y}\right)
\end{aligned} \tag{B.6}$$

C Abelian five-loop diagrams to the hadronic vacuum polarization

In this Appendix we collect the abelian five-loop diagrams to the hadronic vacuum polarization that are included in the scale-setting in next-to-leading order as described in Sect. 5. For completeness, we also list the four-loop diagrams. Numerical values are given for

$$Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \frac{1}{4\pi^2} \sum_{n=0}^{\infty} D_{n+1} \left(\frac{\alpha(Q)}{\pi}\right)^n. \tag{C.1}$$

The diagrams are shown in Fig. 7. Each diagram shown stands for the class of diagrams obtained, when one adds the diagrams where gluons form self-energy type insertions

rather than exchange type topologies. Diagram (V2) also includes the symmetric diagram, where the two inner loops are interchanged. We obtain:

$$D_4^{(\text{IV1})} = C_F(TN_f)^2 \left[\frac{151}{54} - \frac{19}{9}\zeta(3) \right] \quad (\text{C.2})$$

$$D_4^{(\text{IV2})} = C_F(C_F TN_f) \left[-\frac{101}{64} + \frac{3}{2}\zeta(3) \right] \quad (\text{C.3})$$

$$D_5^{(\text{V1})} = C_F(TN_f)^3 \left[-\frac{6131}{972} + \frac{203}{54}\zeta(3) + \frac{5}{3}\zeta(5) \right] \quad (\text{C.4})$$

$$D_5^{(\text{V2})} = C_F(TN_f)(C_f TN_f) \left[\frac{3571}{576} - \frac{59}{8}\zeta(3) + 2\zeta(3)^2 \right] \quad (\text{C.5})$$

$$D_5^{(\text{V3})} = C_F(TN_f)(C_f TN_f) \left[\frac{10199}{3456} - \frac{7}{2}\zeta(3) + \zeta(3)^2 \right] \quad (\text{C.6})$$

Here $T = 1/2$, $C_F = 4/3$ for $SU(3)$ with fermions in the fundamental representation and $T = 1$, $C_F = 1$ for $U(1)$. The sum of these terms yields d_2^{NLO} and d_3^{NLO} in QED (after proper normalization).

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